

3

Electric flux and Gauss's law

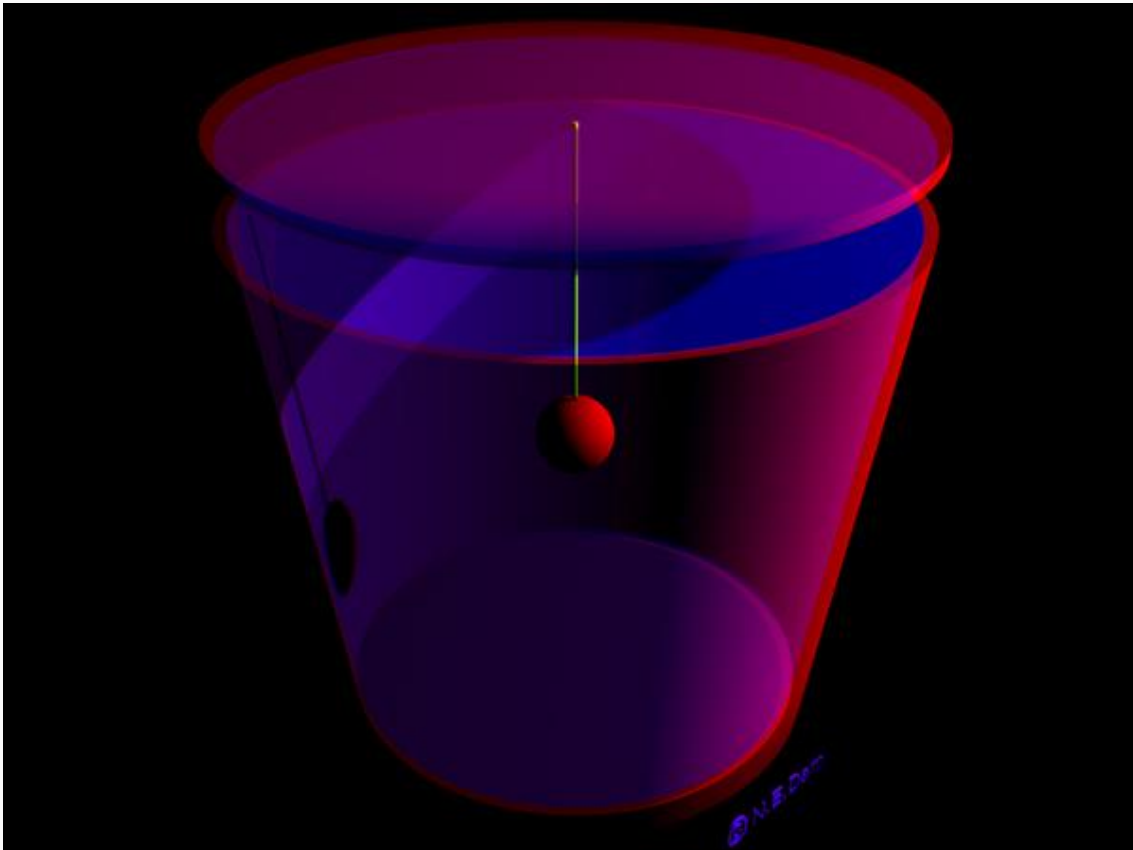


Fig.3.7.8, (N.E. Dam ©). Faradays experiment. A negative charge is induced on the inner surface and a positive charge on the outer surface.

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3.1 Field line density

To begin with, we define that:

The strength of the electric E-field =

The magnitude of the E-field = The length of the E-field vector $E = |\vec{E}|$.

The concept of field lines was introduced in chapter 2. A field line always follows the local electric field. At any point the E-field vector is parallel with the tangent of the E-field line at that point. The density of E-field lines is proportional to the strength of the E-field. Field line is synonymous with flux line.

The density of field lines is the number of field lines passing through an area dA divided by the area itself.

Fig.3.1.1 shows some field lines passing through an area dA . When drawing figures of electric field lines, the density of field lines should be proportional to the strength or magnitude of the electric E-field.

If we are in a region with a constant density of field lines, like in fig.3.1.1, the E-field vector will be constant in direction and magnitude and we say that we have a uniform E-field.

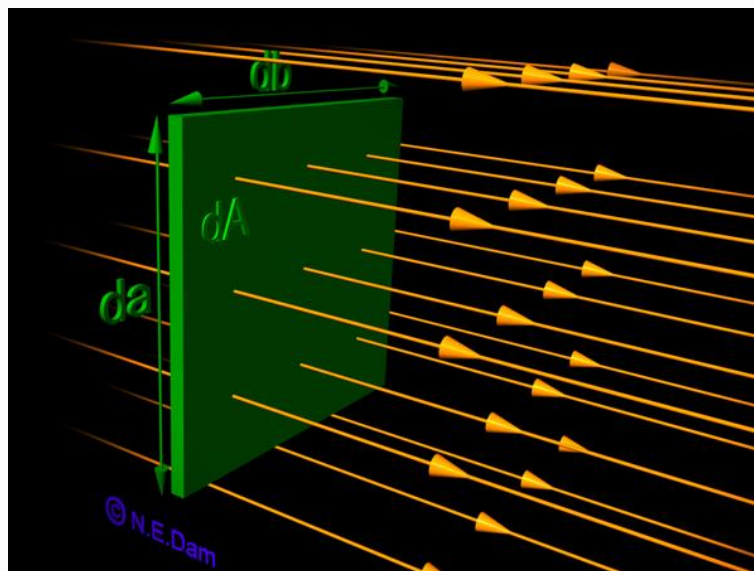


Fig.3.1.1, N.E. Dam ©. E-field lines passing through an area dA . In this region we have a constant density of field lines.

When defining the density of field lines, the area dA should be perpendicular to the field lines. If the area is parallel with the field lines, there will be no field lines passing through the area. We define, that the area should be perpendicular to the field lines. This gives a well defined number of field lines, which is the maximum number of field lines through the area as the orientation of the area is varied.

Fig.3.1.2 shows some field lines around two charges q and $-q$. In this figure, the E-field is not uniform as in fig.3.1.1. We have inserted an area dA at position a and the same area dA at position b. Obviously, we have a higher field line density at point a than at point b. Therefore, the E-field vector is longer at point a than at point b.

Fig.3.1.3 shows that the area dA around point a should not be too large. Otherwise, it would reach regions with a lower field line density. When we calculate the field line density, it is an average value of the field line density across the area dA . Hence, dA should be a relatively small area close to the point where we want to find the density. Besides, the area dA in the figure should be perpendicular to the field lines at least to some approximation.

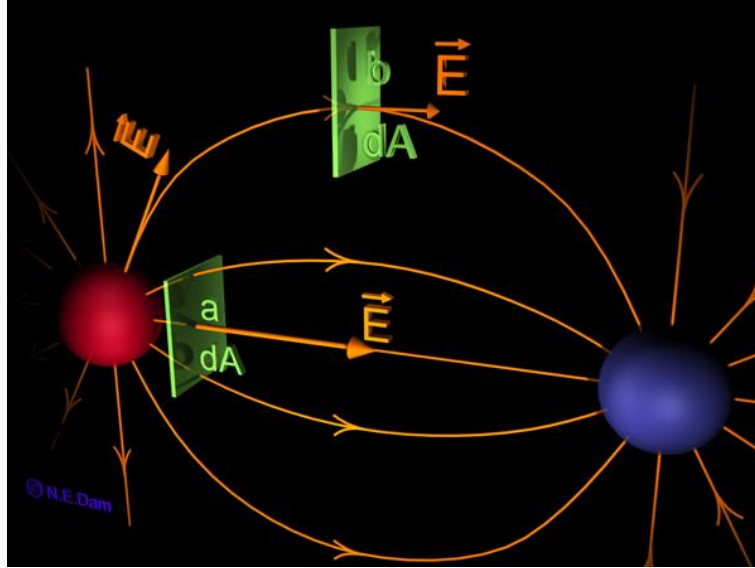


Fig.3.1.2, N.E. Dam ©. Density of E-field lines in a non-uniform E-field at two points a and b.

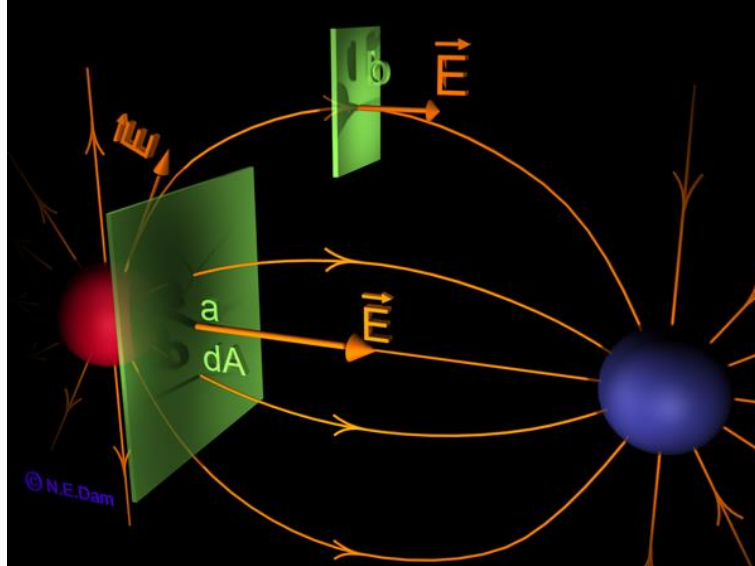


Fig.3.1.3, N.E. Dam ©. If we want to find the density of E-field lines at point a, the area dA around point a should not be too large. Otherwise dA reaches regions with a lower field line density which gives a wrong value for the density close to point a.

Vector-field

In fig.3.1.1 the E-field is uniform because the E-field vector has the same length and direction at all points in the figure. In fig.3.1.2 the E-field is non-uniform because the E-field vector does not have the same length and direction at all points. Each point is given by its coordinates (x,y,z) . No matter if we consider the uniform E-field in fig.3.1.1 or the non-uniform E-field in fig.3.1.2 we have a so-called vector-field.

A vector-field means that we may assign a vector $\vec{v}$ to each point in the considered part of the 2D or 3D-space. In general, the length and the direction of vector $\vec{v}$ depend on which point (x,y,z) we consider as illustrated in fig.3.1.2, see for example point a and point b. Vector $\vec{v}$ has three coordinates:

$$\vec{v} = (v_x, v_y, v_z) \quad \text{eq. 3.1.1}$$

The length and the direction of vector $\vec{v}$ may be calculated from the vector coordinates (v_x, v_y, v_z) . The vector coordinates (v_x, v_y, v_z) may be calculated from the point coordinates (x, y, z) :

$$v_x = v_x(x, y, z) , \quad v_y = v_y(x, y, z) , \quad v_z = v_z(x, y, z) \quad \text{eq. 3.1.2}$$

This means that v_x is a function of (x, y, z) . In other words v_x depends on the point coordinates (x, y, z) . Likewise for the other two vector coordinates v_y and v_z .

$$\vec{v} = (v_x(x, y, z), v_y(x, y, z), v_z(x, y, z)) \quad \text{eq. 3.1.3}$$

In other words, all three coordinates of vector $\vec{v}$ depend on which point (x, y, z) we consider. If the vector field is an electric E-field, the E-field vector is

$$\vec{E} = (E_x(x, y, z), E_y(x, y, z), E_z(x, y, z)) \quad \text{eq. 3.1.4}$$

How to calculate and depict a vector field:

1. We choose a specific point "a" with the coordinates (x_a, y_a, z_a) .
2. The first vector coordinate is a function of the point coordinates, $v_x = v_x(x, y, z)$. When we have chosen a specific point (x_a, y_a, z_a) , these values are substituted into the function $v_x = v_x(x, y, z)$ and we get the first vector coordinate at point "a", $v_{x,a} = v_x(x_a, y_a, z_a)$.
3. In the same way, we calculate the second and the third vector coordinate $v_{y,a} = v_y(x_a, y_a, z_a)$, $v_{z,a} = v_z(x_a, y_a, z_a)$.
4. Now, we know all three coordinates of vector $\vec{v}$ at point "a" and we may draw the vector $\vec{v}$ at the chosen point "a".
5. We choose another point "b" with other coordinates $(x, y, z) = (x_b, y_b, z_b)$.
6. We calculate the three coordinates of vector $\vec{v}$ using the same functions as we used in step 2 and 3, but with other values for (x, y, z) .
7. Now, we know all three coordinates of vector $\vec{v}$ corresponding to point "b": $v_{x,b} = v_x(x_b, y_b, z_b)$, $v_{y,b} = v_y(x_b, y_b, z_b)$, $v_{z,b} = v_z(x_b, y_b, z_b)$.
8. We may continue to calculate and draw vector arrows at many other points (c, d, e, f, g, ...) to establish a picture of the vector field. In practice, a computer program is used for the repeated calculations.

3.2 The concept of flux

Basic definition of flux

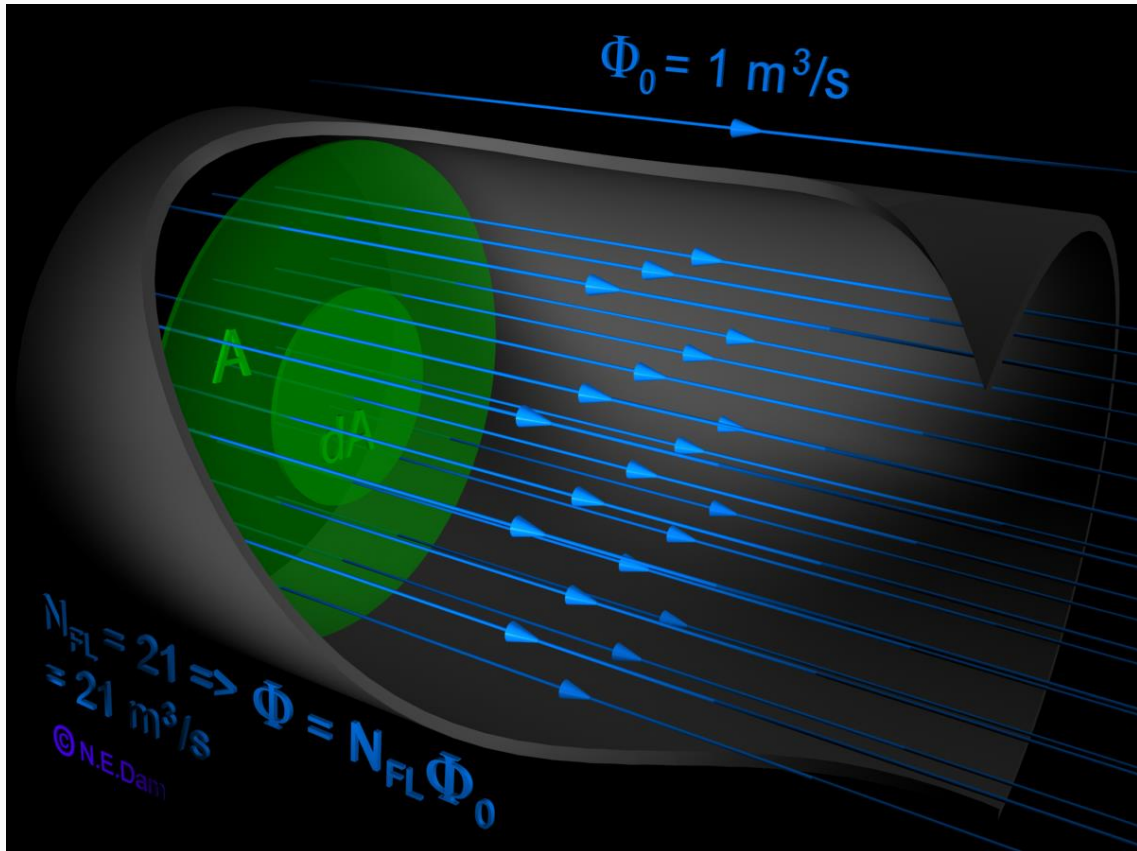


Fig.3.2.1, N.E. Dam ©. Flow of water through a pipeline. The blue lines are the so-called flux lines. Each flux line represents $1 \text{ m}^3/\text{s}$. Counting 21 flux lines, there must be a flow of $21 \text{ m}^3/\text{s}$.

The flux may be defined as the flow of something through an area A or dA .

Flux is denoted by the Greek capital letter Φ . The area A may be an *imaginary* area but it may also be a real surface which is *permeable* to the flow, i.e. the area allows the flow to pass through it.

The concept of flux may be visualized using the flow of water through a pipeline. This is shown in fig.3.2.1.

Let Φ_0 be the amount of flux assigned to one flux line and let N_{FL} be the Number of Flux Lines passing through the area A . Then the total flux through the area A is

$$\Phi = N_{FL} \Phi_0 \quad \text{eq. 3.2.1}$$

In words, the total flux is the number of flux lines times the flux per flux line.

The blue lines in fig.3.2.1 are the so-called flux lines. A is the cross section area of the pipeline. In fig.3.2.1 and in fig.3.2.2 there are 21 flux lines through A . If each flux line represents 1 m^3 of water per second, then $\Phi_0 = 1 \text{ m}^3/\text{s}$. Then the total flux through A is:

$$\Phi = N_{FL} \Phi_0 = 21 \cdot 1 \text{ m}^3/\text{s}$$

If we consider the smaller area dA in fig.3.2.2 , we may count the number of flux lines to be $dN_{FL} = 5$. The amount of flux passing through dA is

$$d\Phi = dN_{FL} \Phi_0 \quad \text{eq. 3.2.2}$$

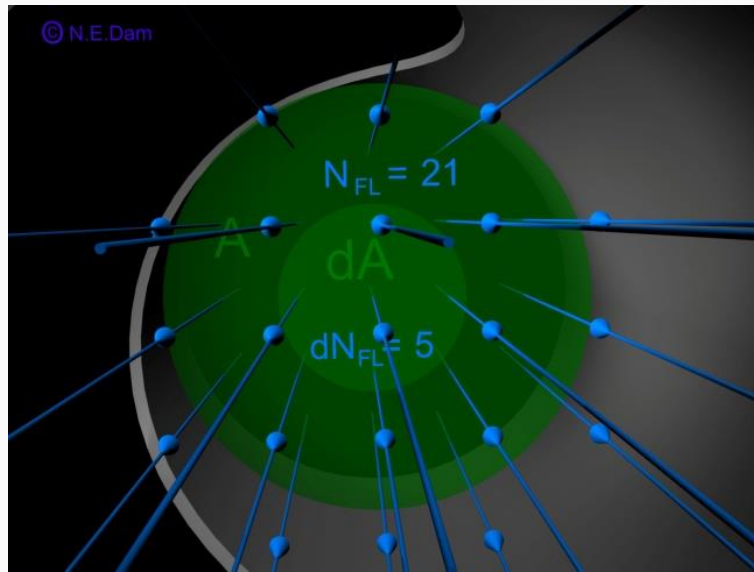


Fig.3.2.2, N.E. Dam ©. Looking in the pipeline direction, it may be easier to count the number of flux lines.

If the pipeline is replaced by an electric wire as shown in fig.3.2.3, we could let:

$$\Phi_0 = 1 \text{ C/s} = 1 \text{ A} . \text{ Then } \Phi = N_{FL} \Phi_0 = 21 \cdot 1 \text{ A}$$

This is the electric current I through the cross section area A .

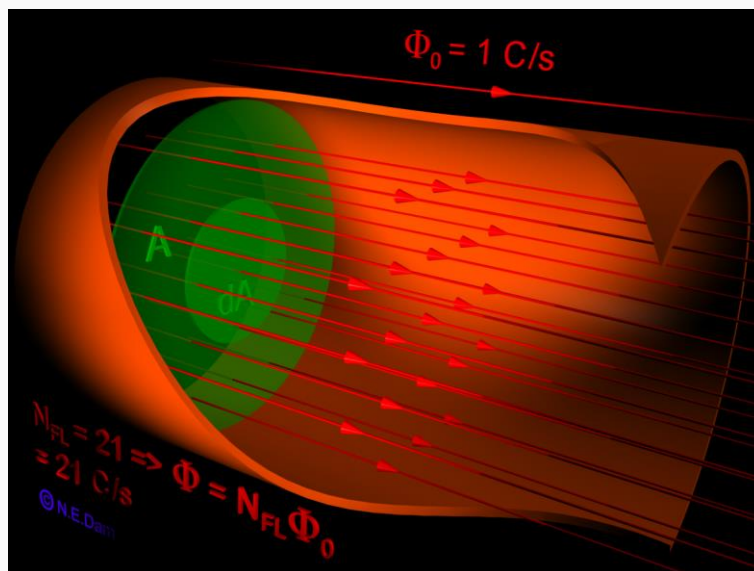


Fig.3.2.3, N.E. Dam ©. Flow of charge through a wire is an example of flux. The red lines are the so-called flux lines. Each flux line represents 1 C/s . Counting 21 flux lines, there must be a flow of $21 \text{ C/s} = 21 \text{ A}$.

Type of flux

If the flux is associated with the vector field $\vec{v}$, we may want to denote the flux as a v-flux:

$$\Phi_v = N_{FL} \Phi_{v0} \quad \text{eq. 3.2.3}$$

The flux per flux line, Φ_{v0} , should correspond to the type and to the amount of flux we consider. E.g. if we consider a flow of water, Φ_{v0} could be $1 \text{ m}^3 \text{ water/s}$ or it could be $0,1 \text{ m}^3 \text{ water/s}$. In the latter case, N_{FL} would be 210 instead of 21 to represent the same amount of flux ($21 \text{ m}^3 \text{ water/s}$) through the cross section A. Hence, Φ_{v0} should be adjusted to give an appropriate number of flux lines in the figure.

The type of flux we have considered so far is a material flux. It has been water or charged particles flowing through an imaginary cross section area A.

However, the flux could also be an immaterial flux such as the E-flux from an E-field. This kind of flux does not involve a flow of a material matter like water or charged particles. It is a flow of an E-vector field through an area A.

The mathematical equations are exactly the same no matter what the type of the flux.

Density of flux

If we divide both sides of eq.3.2.2 by dA , we get the density of flux

$$\frac{d\Phi}{dA} = \frac{dN_{FL}}{dA} \Phi_0 \quad \text{eq. 3.2.4}$$

The flux density equals the density of flux lines times the flux per flux line.

The E-field lines in fig.3.1.1 and in fig.3.1.2 may also be considered as flux lines. The so-called E-flux through dA is denoted by $d\Phi_E$.

Now, we define the E-field magnitude to be the density of E-flux

$$E \equiv \frac{d\Phi_E}{dA} \quad \text{eq. 3.2.5}$$

The E-flux through an area dA is then defined to be

$$d\Phi_E \equiv E dA \quad \text{eq. 3.2.6}$$

We may calculate the amount of E-flux using $E dA$, which is the density of E-flux times the cross sectional area.

If we combine eq.3.2.4 and eq.3.2.5 we find

$$E = \frac{d\Phi_E}{dA} = \frac{dN_{FL}}{dA} \Phi_{E0} \quad \text{eq. 3.2.7}$$

From this we see that the E-field magnitude is the E-flux per area and this is proportional to the number of flux lines per area. Since flux lines are synonymous with field lines, the E-field magnitude is proportional to the density of field lines as mentioned in the beginning of section 3.1.

So far we have assumed that the flux lines are perpendicular to the area dA .

Flux through an area which is not perpendicular to the flux lines

Fig.3.2.4 shows an area $dA = da db$, which is perpendicular to the flux lines. According to eq.3.2.6 the flux may be calculated by

$$d\Phi_E \equiv E dA \quad \text{eq. 3.2.6}$$

as long as the area dA is perpendicular to the flux lines.

If we tilt the area dA by an angle θ , we observe that the number of flux lines passing through the tilted area is reduced relative to the number of flux lines through the vertical area, which is perpendicular to the flux lines. The “new” number of flux lines through the tilted area is the same as the number of flux lines through the part of the vertical area in shadow, when light is coming in the opposite direction of the flux lines. The height of the shadow area is da times $\cos \theta$, hence the shadow area is $db da \cos \theta = da db \cos \theta = dA \cos \theta$. To find the flux through the shadow area, we must multiply E by the shadow area, which is perpendicular to the flux lines.

$$d\Phi_E = E dA \cos \theta \quad \text{eq. 3.2.8}$$

This is the same as the flux through the tilted area.

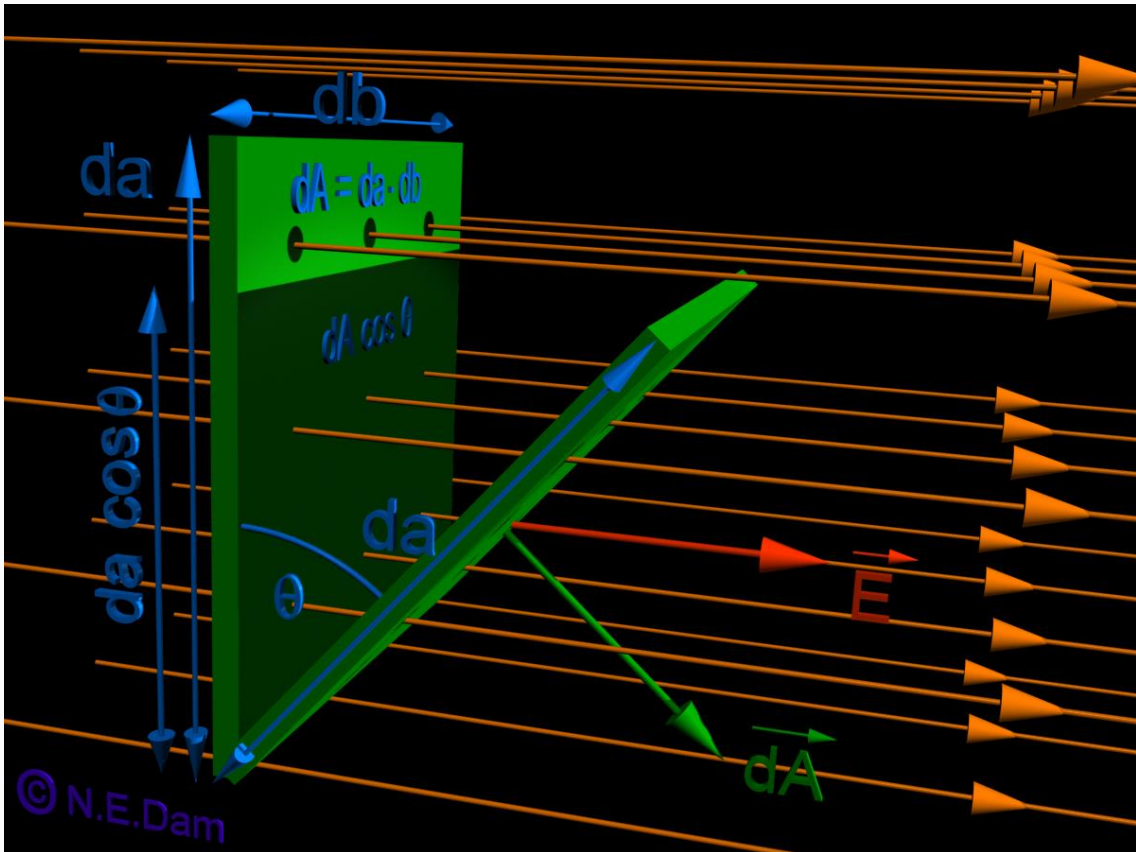


Fig.3.2.4, N.E. Dam ©. Flux lines passing through a tilted area dA .

In fig.3.2.4 we also see the E-field vector $\vec{E}$ and a so-called area vector $\vec{dA}$. The latter is a vector, that is assigned to the tilted area. The magnitude of the vector is simply the area

$$|\vec{dA}| = dA = da db \quad \text{eq.3.2.9}$$

The direction of the vector is perpendicular to the area. The angle θ is from $\vec{E}$ to $\vec{dA}$. Now the flux in eq.3.2.8 may be expressed as the scalar product

$$d\Phi_E = \vec{E} \cdot \vec{dA} \quad \text{eq. 3.2.10}$$

This is the general formula to calculate the E-flux no matter if the area is or is not perpendicular to the flux lines! Some remarks to eq.3.2.10

- The E-flux through an area dA is the scalar product $\vec{E} \cdot \vec{dA}$
- The flux is positive if $\vec{E}$ is to the same side as $\vec{dA}$
- The flux is negative if $\vec{E}$ is to the opposite side of $\vec{dA}$
- The flux is zero if $\vec{E}$ is perpendicular to $\vec{dA}$ meaning that the E-field lines are parallel with the surface of the area dA .

What we do is in fact the following. The tilted area is projected in the opposite direction of the flux lines. Then we find the flux through the projected area, which is perpendicular to the flux lines.

Flux through a curved area in a non-uniform field

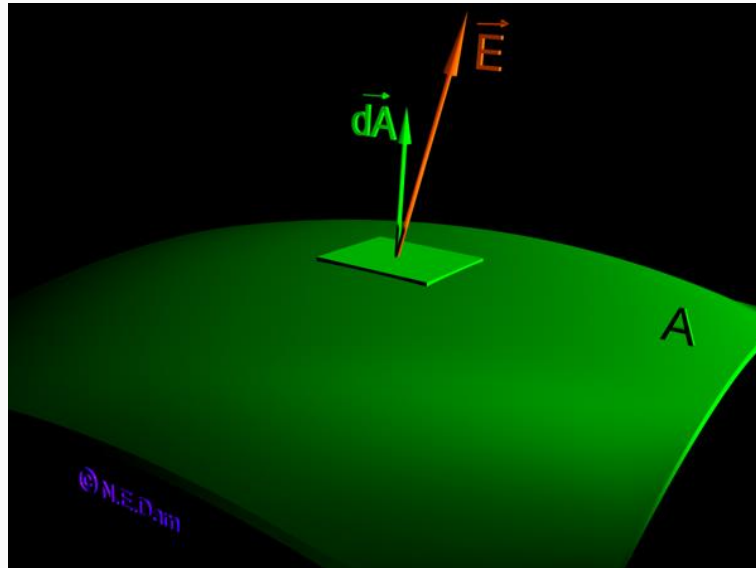


Fig.3.2.5, N.E. Dam ©. An area dA which is a small part of a curved area A .

The light green area in fig.3.2.5 is a small flat area dA , which is tilted relative to E-field direction just like the tilted area dA in fig.3.2.4. The flux through the area dA is

$$d\Phi_E = \vec{E} \cdot \vec{dA} \quad \text{eq. 3.2.10}$$

Here, we assume, that the E-field is almost constant (uniform) across the small flat area dA . If A is a flat area and the E-field is uniform, the flux through A may be calculated this way:

$$\Phi_E = \vec{E} \cdot \vec{A} \quad \text{eq. 3.2.11}$$

If the larger area A is curved or the E -field is not uniform, we may calculate the flux through the small area dA using eq.3.2.10 and then integrating the flux through all small areas dA across A :

$$\Phi_E = \int_A d\Phi_E = \int_A \vec{E} \cdot \vec{dA} \quad \text{eq. 3.2.12}$$

where we integrate over the curved area A .

This formula is the general formula to find the flux through an area A .

It should be used if the area A is curved or if the E -field is varying in direction or magnitude across the area A .

Please note, that $\vec{dA}$ defines a positive direction for passing the area or surface. It is up to us to decide the positive direction $\vec{dA}$. It should just be perpendicular to the dA surface. In fig.3.2.4 and in fig.3.2.5 the $\vec{E}$ vector is not in the same direction as $\vec{dA}$. However, the $\vec{E}$ vector is to the same side as $\vec{dA}$. Hence, the scalar product in eq.3.2.10, which is the E -flux, will be positive. If the $\vec{E}$ vector is perpendicular to $\vec{dA}$, the flux will be zero. If the $\vec{E}$ vector points to the opposite side of $\vec{dA}$, the flux will be negative.

3.3 The flux through a closed surface

If we have a closed surface, the area vector $\vec{dA}$ should always point out of the closed surface A . Hence, the flux will be positive when the flux lines are coming out of the closed surface. To indicate, that we integrate over a closed surface, we draw a small circle in the integral sign. Now we get for the flux out of a closed surface:

$$\Phi_E = \oint_A \vec{E} \cdot \vec{dA} \quad \text{eq. 3.3.1}$$

Fig.3.3.1 shows the most simple closed surface. It is a sphere. In fact we see two spheres. In the centre we have a positive charge q . Now we want to find the flux out of the larger sphere of radius R_2 .

We use eq.3.3.1 to find the flux. The figure shows one of the small area elements $\vec{dA}$. As it might be seen, the field vector $\vec{E}$ is parallel with $\vec{dA}$ because the field lines or flux lines are directed radially away from q in the centre. Hence

$$\vec{E} \cdot \vec{dA} = E dA \quad \text{eq. 3.3.2}$$

This simplifies the integration greatly. From chapter 2 we have for the E -field:

$$E = \frac{q}{4\pi\epsilon_0 r^2} \quad \text{eq. 3.3.3}$$

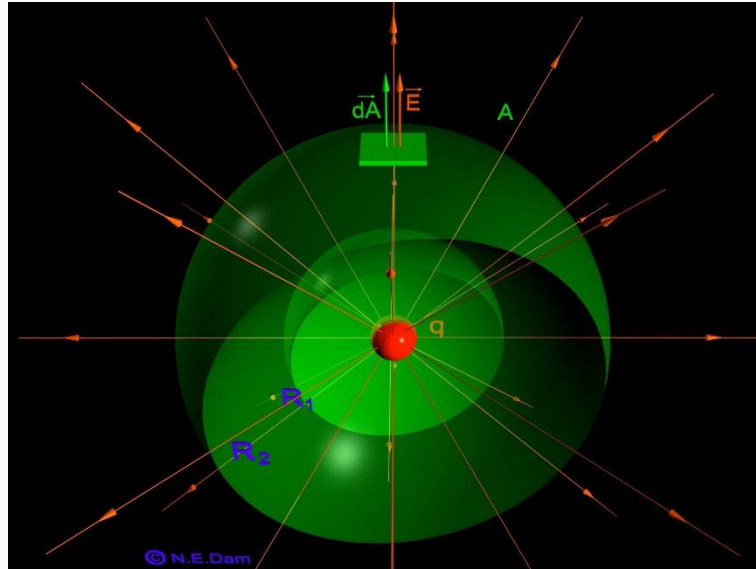


Fig.3.3.1, N.E. Dam ©. The flux out of two spheres of radius R_1 and $R_2 > R_1$.

At the surface of the large sphere we have $r = R_2$. Hence E must be constant at the surface. It simplifies the integration, since E can be moved outside the integral:

$$\Phi_E = \oint_A \vec{E} \cdot d\vec{A} = \oint_A E dA = E \oint_A dA = \frac{q}{4\pi\epsilon_0 R_2^2} 4\pi R_2^2 = \frac{q}{\epsilon_0} \quad \text{eq. 3.3.4}$$

where the last integral over the large sphere is the area of the large sphere, which is $4\pi R_2^2$. We now see, that the flux is independent of the radius of the sphere! The flux out of the small sphere of radius R_1 will be the same as for the large sphere of radius R_2 . It should be because we have the same flux lines and hence the same number of flux lines out of the two spheres.

If we deform the larger sphere in fig.3.3.1 to another form like in fig.3.3.2, we get the same flux as before with the sphere and q in the centre.

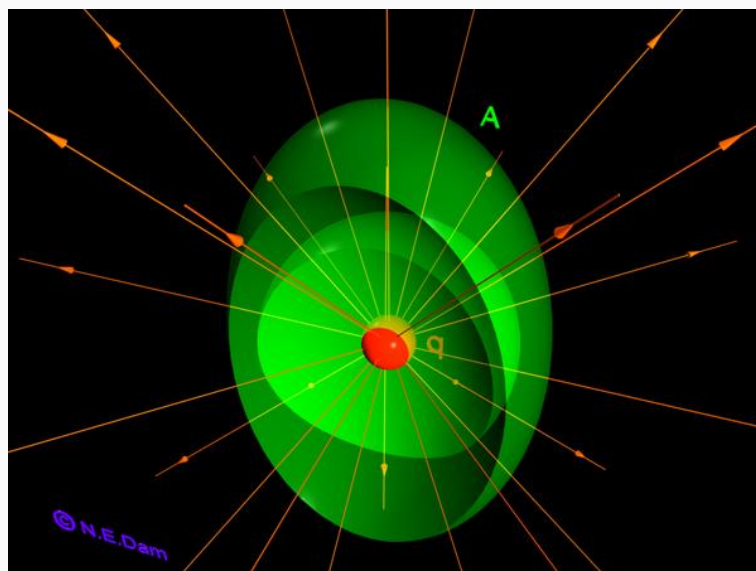


Fig.3.3.2, N.E. Dam ©. The flux out of a deformed closed surface.

The charge q does not have to be a point charge. It is the same flux lines going out of the inner sphere and the outer deformed surface, also if the outer surface is displaced, so q is not in the centre, as long as q is inside both surfaces.

$\vec{E}$ will only be parallel with $\vec{dA}$ at a few points on the deformed surface and E will not be constant at different points on the surface. Hence, the integral cannot be simplified as was the case with the charge q in the centre of the sphere.

3.4 Gauss's law

Gauss's law for a single charge q

As mentioned above, the flux out of a closed surface enclosing a charge q will be independent of the size and the form of the closed surface. The flux is also independent of the position and the extension of the object carrying the charge q as long as q is inside the closed surface. The result from eq.3.3.4 is:

$$\Phi_E = \oint_A \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0} \quad \text{eq. 3.4.1}$$

This is Gauss's law for a single charge q . In words it is:

- The total E-flux out of a closed surface A is the integral over A of $\vec{E} \cdot d\vec{A}$
- The total E-flux is also equal to the enclosed charge q divided by ϵ_0 .

Flux lines, that leaves and enters the same closed surface

Fig.3.4.1 shows the E-field lines from two charges q and $-q$ enclosed by a surface. On the lower side of the surface, the E-field is pointing outwards and here we have a positive flux out of the surface. On the upper side of the surface, the E-field is pointing inwards and here we have a negative flux. The sum of the positive flux and negative flux is zero. This is a consequence of Gauss's law for a single charge as well as the superposition principle (for details see below). If we count the number of flux lines, there will be as many flux lines pointing outwards as inwards. This shows that we have the same amount of flux out of the closed surface as we have flux into the closed surface.

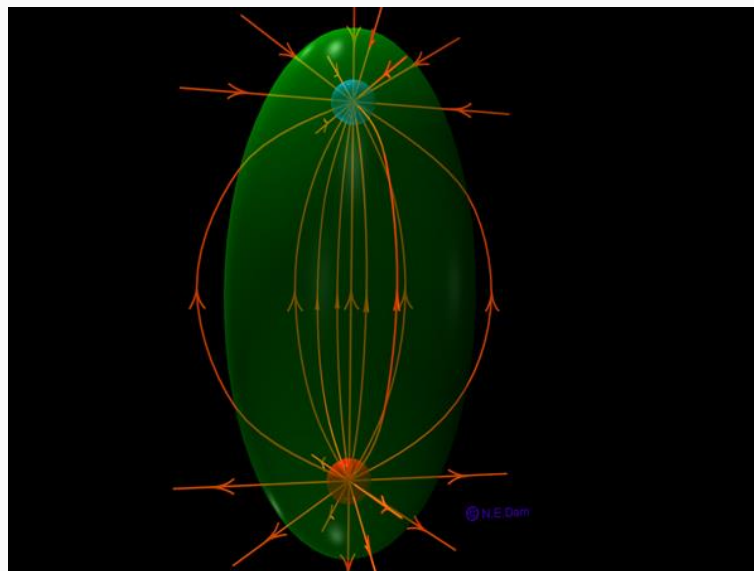


Fig.3.4.1, N.E. Dam ©. The flux out of a closed surface with two charges q and $-q$.

Fig.3.4.2 shows a situation where we have a charge q outside the closed surface. Below we will show that the total flux through the closed surface is $\Phi_E = 0$. The figure shows that all the flux lines going into the surface also come out of the surface. It will be explained in more detail.

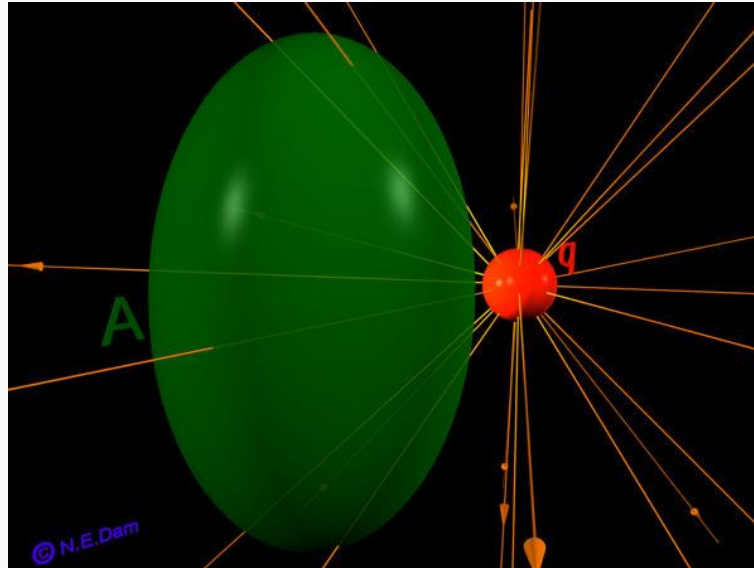


Fig.3.4.2, N.E. Dam ©. The flux out of a deformed closed surface with no charges inside.

According to eq.3.2.10 and the remarks below this equation, the flux through a surface element dA is positive or negative depending on the E-field vector points to the same side or to the opposite side of vector $\vec{dA}$.

By definition, the $\vec{dA}$ vector always points outwards from a closed surface. Hence, we get a positive flux at surface points, where the E-field vector also points outwards from the closed surface. This means that $\vec{E}$ is to the same side as $\vec{dA}$ at the surface point. The E-field vector does not have to be perpendicular to the surface.

Likewise, we get a negative flux at surface points, where the E-field vector points into the closed surface at some angle.

Now, we consider the flux lines from q in fig.3.4.2. Most of the flux lines pass beside the closed surface and will not generate any flux through the closed surface. The remaining flux lines pass through the closed surface.

Now, we follow one of the flux lines passing through the closed surface.

Such a flux line generates a negative flux where it enters the closed surface because the E-field vector points into the closed surface at that point. The same flux line passes straight through the closed surface and comes out of the closed surface, where it generates a positive flux because the E-field vector points out of the closed surface at that point. According to the basic definition of flux in eq.3.2.1 or eq.3.2.2, the flux through a surface is proportional to the number of flux lines. Therefore, when we consider one single flux line, that enters and leaves the same closed surface, the negative flux will cancel with the positive flux.

The result is, that each flux line, that enters and leaves the closed surface, will not contribute with any flux through the closed surface.

No flux lines start or stop inside the surface. This would be the case if there was a charge inside the surface. The flux out of the surface is only affected by the charge inside the surface!

Gauss's law in the general form with more charges

Below, we derive Gauss's law in the general form, where we have more charges inside and outside the closed surface. We consider an example with a positive and a negative charge inside the closed surface and a positive charge outside the closed surface. In fig.3.4.3 we have a positive charge q_1 (red) and a negative charge q_2 (blue) at different points within a closed surface. We also have a third charge q_3 (orange) outside the closed surface.

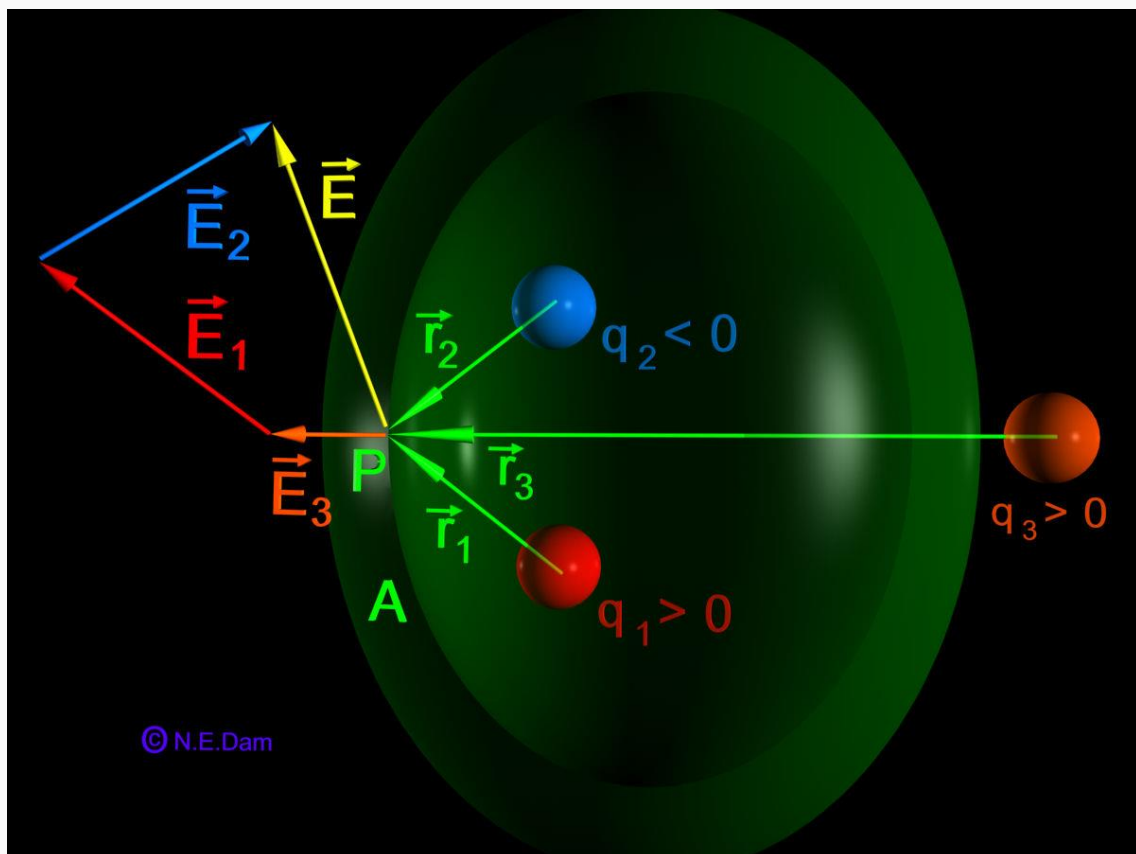


Fig.3.4.3, N.E. Dam ©. When calculating the flux out of a closed surface we should know the E-field at any point P on the surface. This E-field is a vector sum, a superposition, of the E-fields from all the charges inside and outside the closed surface. In order to see what is inside the closed surface, we have cut out a part of it.

We calculate the E-flux from the three charges:

Contribution from q_1

If we “turn off” q_2 and q_3 , then q_1 is the only “active” charge. It generates the E-field $\vec{E}_1$ at the surface point P. In fig.3.4.3 vector $\vec{E}_1$ is drawn a little away from P because it becomes a part of a sum. We return to that later. If only q_1 is active, $\vec{E}_1$ is the only E-

field. We should imagine, that $\vec{E}_1$ begins at point P and points radial away from q_1 because q_1 is positive.

According to eq.3.4.1, Gauss's law for q_1 is:

$$\Phi_{E1} = \oint_A \vec{E}_1 \cdot \vec{dA} = \frac{q_1}{\epsilon_0} \quad \text{eq. 3.4.2}$$

Contribution from q_2

If we “turn off” q_1 and q_3 , then q_2 is the only “active” charge. It generates the E-field $\vec{E}_2$ at the surface point P. In fig.3.4.3 vector $\vec{E}_2$ is drawn away from P because it becomes a part of a sum. If only q_2 is active, $\vec{E}_2$ is the only E-field. We should imagine, that $\vec{E}_2$ begins at point P and points towards q_2 because q_2 is negative.

According to eq.3.4.1, Gauss's law for q_2 is:

$$\Phi_{E2} = \oint_A \vec{E}_2 \cdot \vec{dA} = \frac{q_2}{\epsilon_0} \quad \text{eq. 3.4.3}$$

Contribution from q_3

If we “turn off” q_1 and q_2 , then q_3 is the only “active” charge. Note, that q_3 is outside the closed surface. The charge q_3 generates the E-field $\vec{E}_3$ at the surface point P. In fig.3.4.3 vector $\vec{E}_3$ begins at point P and points radial away from q_3 if q_3 is positive.

When q_3 is the only active charge and q_3 is placed outside the closed surface, the situation is the same as described in fig.3.4.2 . A part of the flux from q_3 flows through the closed surface A. The flux from q_3 flowing through A is zero because all the flux lines that enter the closed surface A also come out of the surface. The flux flowing through the closed surface A is calculated using the same kind of integral as if q_3 was inside the closed surface A, i.e. it is an integral over the closed surface A of $\vec{E}_3 \cdot \vec{dA}$. We just know, that the total flux through A coming from q_3 will be zero. Therefore, we have

$$\Phi_{E3} = \oint_A \vec{E}_3 \cdot \vec{dA} = 0 \quad \text{eq. 3.4.4}$$

Addition of the contributions from q_1 , q_2 , and q_3

To find the total flux we use the principle of superposition. We add eq.3.4.2, eq.3.4.3 and eq.3.4.4 and find

$$\begin{aligned} \Phi_{E \text{ tot}} &= \Phi_{E1} + \Phi_{E2} + \Phi_{E3} = \\ \oint_A (\vec{E}_1 + \vec{E}_2 + \vec{E}_3) \cdot \vec{dA} &= \oint_A \vec{E} \cdot \vec{dA} = \frac{q_1 + q_2 + 0}{\epsilon_0} = \frac{q_{\text{encl}}}{\epsilon_0} \quad \text{eq. 3.4.5} \end{aligned}$$

where q_{encl} is the sum of the charges inside the closed surface (q_1 and q_2). Vector $\vec{E}$ without subscript is the total E-field at point P on the surface as illustrated in fig.3.4.3. Note, that all three charges contributes to the E-field at the surface point P, see $\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3$ in the integral. At the same time it is only the two charges inside the closed surface, that contributes to the enclosed charge on the right side, $q_{\text{encl}} = q_1 + q_2$. The reason for that is eq.3.4.4, where $\vec{E}_3$ occurs on the left side while q_3 does not occur on the right side.

The general form of Gauss's law

However, eq.3.4.5 may be generalized to more than two charges. Then q_{encl} is the sum of all charges within the closed surface, i.e. the total amount of charge enclosed by the surface. This leads to the general form of Gauss's law:

$$\Phi_E = \oint_{\text{GS}} \vec{E} \cdot \vec{dA} = \frac{q_{\text{encl}}}{\epsilon_0} \quad \text{eq. 3.4.6}$$

Notes concerning eq.3.4.6:

- The closed surface A is now denoted as the so-called Gaussian Surface (GS).
- When we have a closed surface, the vector $\vec{dA}$ should always point out of the closed surface. This implies that the flux is positive at surface points where the $\vec{E}$ vector is pointing out of the closed surface. An outward flux is positive, while an inward flux is negative.
- In fig.3.4.3 q_1 and q_2 are arbitrary. If $q_2 = -q_1$, the sum $q_{\text{encl}} = 0$. Then the total electric flux out of the surface is $\Phi_E = 0$. The E-field inside the integral in eq.3.4.6 is the total E-field at the surface. It is not necessarily zero when $q_{\text{encl}} = 0$ on the right side of eq.3.4.6. Fig.3.4.2 shows clearly, that even if $q_2 = -q_1$, and thereby $q_{\text{encl}} = 0$, we have an electric field at the surface. Hence, we may have an E-field at surface points even if the total flux out of the closed surface is zero.

3.5 The electrostatic condition

The electrostatic condition means that the charges must not move through the material. If a change occurs in the charge distribution, there will be a time interval, a relaxation time, where the charges redistribute.

The electrostatic condition inside an insulator

Inside an insulator, the charges cannot move through the material since all the positive nuclei and all the negative electrons are bound to the molecules. The charges cannot move from one molecule to another. It would require an extremely strong electric field and electric force to pull some of the electrons out of the molecules and the material would be destroyed. This phenomenon is called electric breakdown. Under normal conditions, the electric field is not extremely strong, and the electrons cannot move through the material. In other words:

No matter if we have an electric field or not inside an insulator the electrostatic condition is fulfilled.

The glass and plastic rods described in chapter 2 are only charged on the outer surface. Inside, the material is neutral.

Normally, we have no excess charge inside an insulator but there may be excess charge on the outer surface.

If we have an accelerator, it is possible to fire ions into an insulating material. This is the kind of material used for the insulating sphere in section 3.6.3. In this example, the ions are assumed to be uniformly distributed inside the material as shown in fig.3.6.3. This situation is static. The charged ions will create an E-field inside the material and this E-field creates forces on all positive and negative charges inside the material. However, none of the charges will start to move because charge cannot move through an insulator.

Inside a conductor

First we have to explain briefly how charge can move inside a conductor.

Inside a conductor, some of the electrons, the so-called conduction electrons, are able to move almost freely. In a metal crystal structure, each of the atoms deliver a few (very often just one) conduction electron to the common metal crystal structure. Such a conduction electron is able to move almost freely through the metal crystal structure. The conduction electrons are “riding” on the equipotential surfaces around the “fixed” atoms. The only resistance is when the conduction electron, every now and then, collides with an electron belonging to an atom at a fixed position in the structure. The moving conduction electron will then lose some of its energy to the “fixed” electron. This energy is transferred to the “fixed” atom causing the nucleus to vibrate more than before and thereby increasing the temperature in the material.

If we have an electric E-field inside a conductor, this field will be equivalent to an electric force on all charges. However, it will only cause the conduction electrons to

move through the material. The other electrons and all the nuclei will not move through the material. They are bound to the crystal structure.

It is therefore obvious, that we cannot have an electric field inside a conductor if the charge, that is able to move (the conduction electrons), must not move.

The electrostatic condition for a conductor (typically a metal) is therefore, that the electric E-field inside the conductor should be zero.

No excess charge inside a conductor

In this section we will see how the electrostatic condition (no E-field inside the conductor) implies that there is no excess charge inside the conductor:

- No E-field inside the conductor =>
- No excess charge inside the conductor =>
- All excess charge resides on the surface of the conductor

Fig.3.5.1 shows a section of a conductor. It could be a copper wire. We assume electrostatic conditions. Hence we have no E-field inside the conductor.

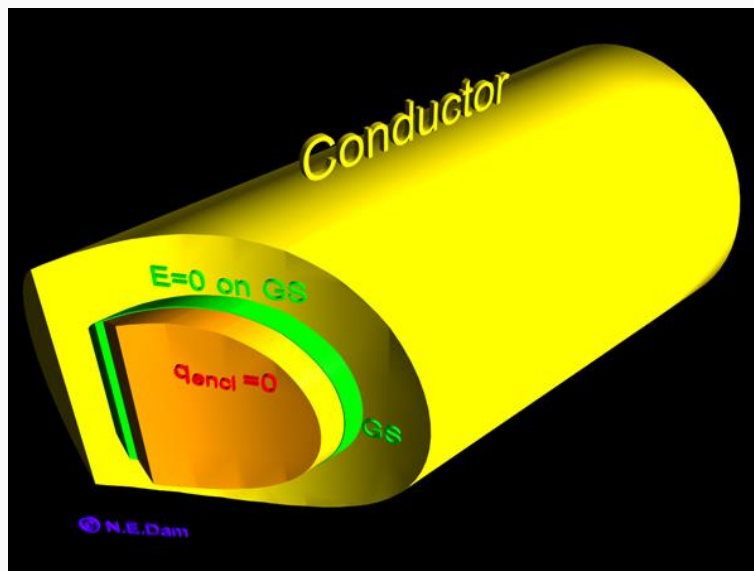


Fig.3.5.1, N.E. Dam ©. The flux out of the closed Gaussian surface (green GS) is zero. Therefore we have no net charge inside the GS.

We now place a closed Gaussian surface GS (green) inside the conductor and calculate the flux out of the GS using Gauss's law eq.3.4.6

$$\Phi_E = \oint_{GS} \vec{E} \cdot d\vec{A} = \frac{q_{encl}}{\epsilon_0} \quad \text{eq. 3.4.6}$$

Since $E=0$ everywhere inside the conductor, the integral must be zero. Hence the right side must also be zero, hence $q_{encl} = 0$.

This must be true inside any GS no matter how big the GS is as long as it is inside the conductor.

The fact that we have no excess charge inside a conductor under electrostatic conditions means that if there is a net charge on the conductor, it will reside on the surface as shown in fig.3.5.2. The excess charge consists of a lot of small charges of the same sign. These charges repel each other and try to maximize their internal distance.

Later, in section 3.7, there is a description of the so-called Faraday cage. Here, we see how an external E-field may redistribute the charge on the surface of a metal box. Below, we repeat the electrostatic condition step by step:

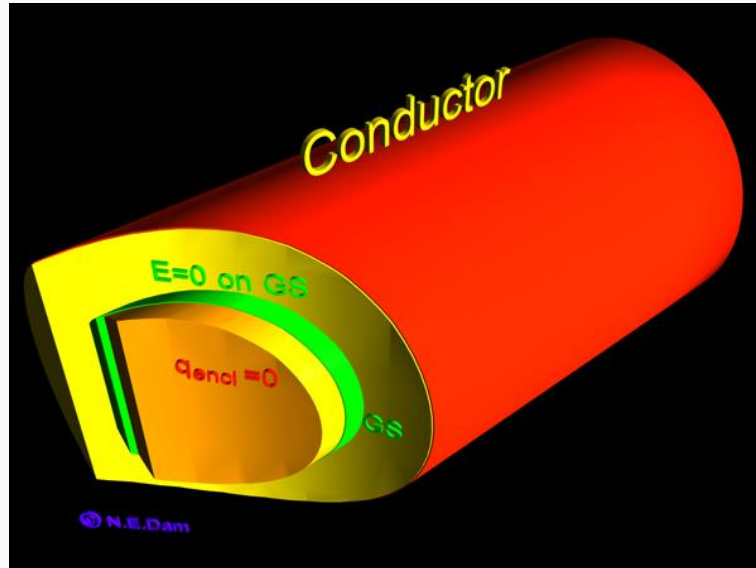


Fig.3.5.2, N.E. Dam ©. If there is a net charge on the conductor, it will reside on the surface.

The electrostatic condition (eq.3.5.1-3.5.4)

1. The conduction electrons must not drift $\vec{F}_{CE} = q\vec{E} = 0 \Rightarrow$
2. The E-field inside the conductor $\vec{E} = 0 \Rightarrow$
3. The integral over the GS $\oint_{GS} \vec{E} \cdot d\vec{A} = 0 \Rightarrow$
4. The enclosed (excess) charge $\frac{q_{encl}}{\epsilon_0} = 0 \Rightarrow q_{encl} = 0$

3.6 Calculation of E-field using Gauss's law

Introduction

In chapter 2 it was described how to calculate the E-field around charged objects of different forms using superposition of the Coulomb fields from small charge elements. Very often this implies solving complicated integrals.

In many cases, the charged object has a high degree of symmetry (e.g. it could be a plane, a cylinder or a sphere). In such cases, the E-field around the object may be calculated using Gauss's law instead of the method described in chapter 2 (superposition of Coulomb fields from small charge elements). As we will see later in this section, the E-field calculation using Gauss's law is much easier to perform than the method using integration or superposition of Coulomb fields.

Gauss's law can be used in two ways:

1. *The first way is to calculate the E-field, if we know the charge distribution, and if the Gaussian surface has enough symmetry to be integrated easily: charge distribution $\Rightarrow$ field*

In this case we use the fact that the electrostatic excess charge only resides on the conductor surface, not in the interior of the conductor as stated above. Calculation of the field is demonstrated later in this section.

2. The second way is to calculate the charge distribution knowing the E-field. charge distribution $\Leftarrow$ field

This will be demonstrated in section 3.7. Here, we use the electrostatic condition, that the E-field is zero inside a conductor. It enables us to deduce where the charge will be located on an inner or outer surface of a conductor.

Solving problems, one must choose a Gaussian surface (GS) with a symmetry corresponding to the charge distribution (a plane, a cylinder or a sphere). The closed surface may be divided into separate surfaces such as sides and ends of a cylinder or box. The best way to consider this is to look at some important examples.

Field of a charged conducting sphere

We want to find the field at any point inside and outside the conducting solid sphere shown in fig.3.6.1. We know that the charge is uniformly distributed on the spherical surface. In the figure, we see a cross section through the solid sphere. The sphere is orange inside the material (no excess charge). Outside it is covered with a red layer of charge. The field has spherical symmetry and it is directed radially away from the sphere with a magnitude, that only depends on the distance r from the centre of the sphere. In order to calculate the field we choose a Gaussian surface (GS) having the same form and symmetry as the object. In this case the GS (green in the figure) should be a concentric sphere placed around the conducting sphere. The word concentric means, that the two spheres have the same centre.

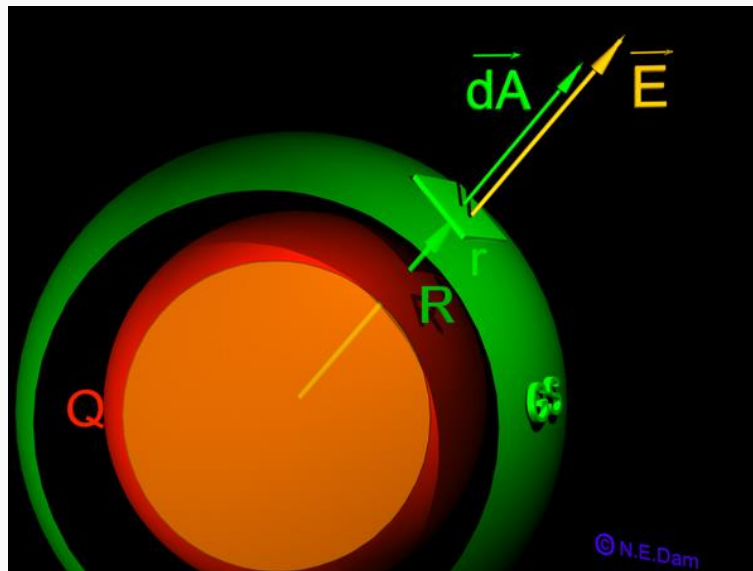


Fig.3.6.1, N.E. Dam ©. Calculation of the E -field inside and outside a conducting sphere.

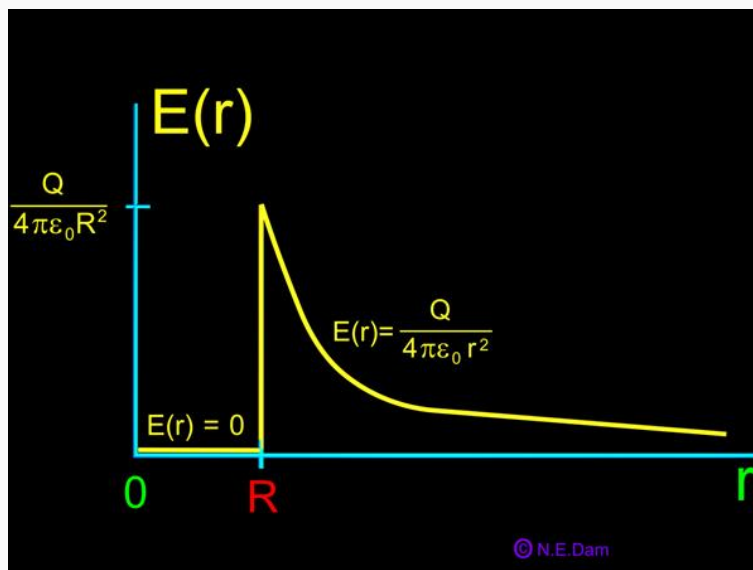


Fig.3.6.2, N.E. Dam ©. The E -field inside and outside a conducting sphere.

Inside the conducting sphere

The E-field is zero according to the electrostatic condition, see the graph in fig.3.6.2 for $0 \leq r \leq R$

Outside the conducting sphere

The E-field is perpendicular to the surface. Hence

$$\vec{E} \parallel \vec{dA} \Rightarrow \vec{E} \cdot \vec{dA} = E dA \quad \text{eq. 3.6.1}$$

Besides, the E-field is constant on the GS where we have a constant distance r to the centre. Hence, E can be moved outside the integral. Thereby the integral becomes the area of the GS, which is $4\pi r^2$. We obtain the following calculation

$$\Phi_E = \oint_{GS} \vec{E} \cdot \vec{dA} = \oint_{GS} E dA = E \oint_{GS} dA = E 4\pi r^2 = \frac{Q}{\epsilon_0} \quad \text{eq. 3.6.2} \Rightarrow$$

$$E = \frac{Q}{4\pi\epsilon_0 r^2} \quad \text{eq. 3.6.3}$$

like the E-field from a point charge, see the graph in fig.3.6.2 for $R < r$.

Try to imagine, how difficult it would be to calculate the field around a sphere by superposition of the E-fields from charge elements on a spherical surface!

Field of a charged insulating sphere

Now, we want to find the E-field inside and outside an insulating sphere of charge Q . Inside an insulator, the charge is assumed to be uniformly distributed all over the entire volume. The insulator is the purple sphere of radius R shown in fig.3.6.3. In the cross section we see that the inner part of the insulator has the same colour as the outer surface of the insulator. It symbolizes that the charge is uniformly distributed inside the insulator.

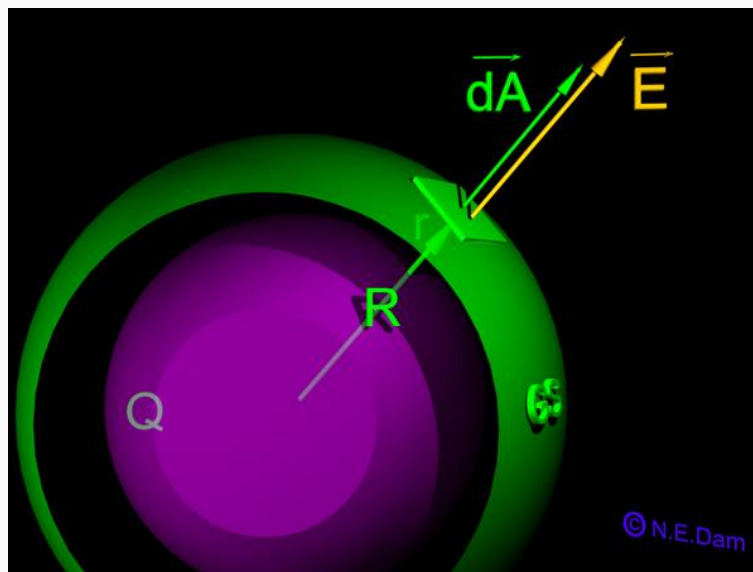


Fig.3.6.3, N.E. Dam ©. Calculation of the E-field outside an insulating sphere.

The charge density is the total charge divided by the total volume of the insulator

$$\rho = \frac{Q}{\frac{4}{3}\pi R^3} \quad \text{eq. 3.6.4}$$

Outside the insulating sphere ($R < r$)

We want to find the E-field at a distance r from the centre of the insulator. We therefore place a (green) spherical Gaussian Surface (GS) of radius r as shown in fig.3.6.3. The GS is concentric with the spherical insulator. The calculation of the E-field outside the insulator is a repetition of the calculation of the E-field outside a conducting sphere, see eq.3.6.2. In this calculation Q is the total enclosed charge inside the GS. In the calculation it does not matter if the charge resides on the outer surface of the sphere (as with the conductor) or resides inside the sphere (as with the insulator). The enclosed charge is exactly the same in both cases. Therefore, we get the same E-field as derived for the conducting sphere in eq.3.6.3. The result is shown in fig.3.6.5 for $R < r$.

Inside the insulating sphere ($r < R$)

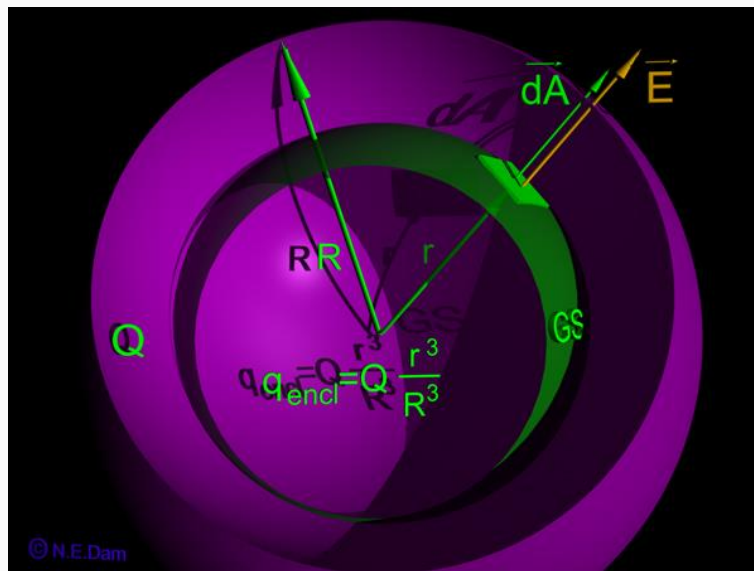


Fig.3.6.4, N.E. Dam ©. Calculation of the E-field inside an insulating sphere.

We place a GS inside the insulator, see the green sphere of radius r in fig.3.6.4.

Because of the spherical symmetry, the E-field has the same magnitude at every point on the GS and it is directed radially away from the centre if $Q > 0$.

The GS of radius $r < R$ does not enclose all the charge inside the insulator but only a fraction of the total charge Q . The enclosed charge is

$$q_{\text{encl}} = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3} Q = \frac{r^3}{R^3} Q \quad \text{eq. 3.6.5}$$

The fraction is the ratio between the volume of the GS (radius r) and the volume of the insulator (radius R). We may also calculate the enclosed charge as the volume density of charge (from eq.3.6.4) times the volume inside the GS.

$$q_{\text{encl}} = \rho \frac{4}{3}\pi r^3 = \frac{r^3}{R^3} Q \quad \text{eq. 3.6.6}$$

For $r < R$, Gauss's law gives

$$\Phi_E = \oint_{\text{GS}} \vec{E} \cdot d\vec{A} = E \oint_{\text{GS}} dA = E 4\pi r^2 = \frac{r^3}{R^3} \frac{Q}{\epsilon_0} \quad \text{eq. 3.6.7} \Rightarrow$$

$$E = \frac{Q}{4\pi\epsilon_0 R^3} r \quad \text{eq. 3.6.8}$$

The E-field inside the insulator, where $r < R$, is a straight line through (0,0) and with the slope $\frac{Q}{4\pi\epsilon_0 R^3}$ as shown in fig.3.6.5. Inside the insulator, the E-field is not the same as for the conducting sphere.

At the surface of the insulating sphere ($r = R$)

If $r = R$ we get the same value for the E-field using both expressions for E (eq.3.6.8 inside and eq.3.6.3 outside the insulator). E is continuous across the surface of the insulating sphere as shown in fig.3.6.5. It is in contrast to the conducting sphere, where the E-field is discontinuous across the surface at $r = R$ as shown in fig.3.6.2.

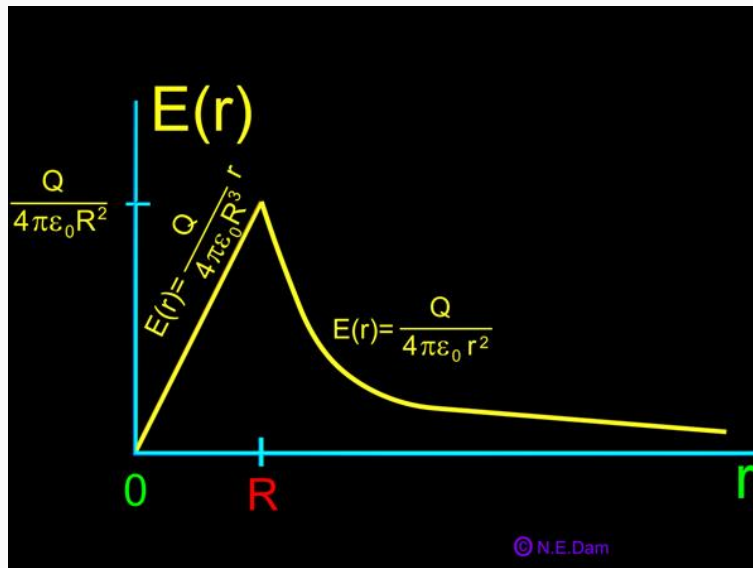


Fig.3.6.5, N.E. Dam ©. The E-field inside and outside an insulating sphere.

Field around a single layer of charge

We want to find the field of an infinite plane sheet of charge. In fig.3.6.6, the plane is not infinite in order to enhance the 3D-effect.

Because of the symmetry and because the sheet is infinite, the E-field vector is perpendicular to the sheet and points away from the sheet on both sides. On the right side of the sheet, we have a flux flow to the right. On the left side of the sheet, we have a flux flow to the left. The E-field radiates away from the sheet, and the sheet is a symmetry plane (or mirror plane) for the E-field.

We use the green box shown in fig.3.6.6 as a GS. Note, that the box passes through the charged sheet. The surface charge density on the sheet is σ . Hence, the enclosed charge on the sheet, but inside the box, is σA .

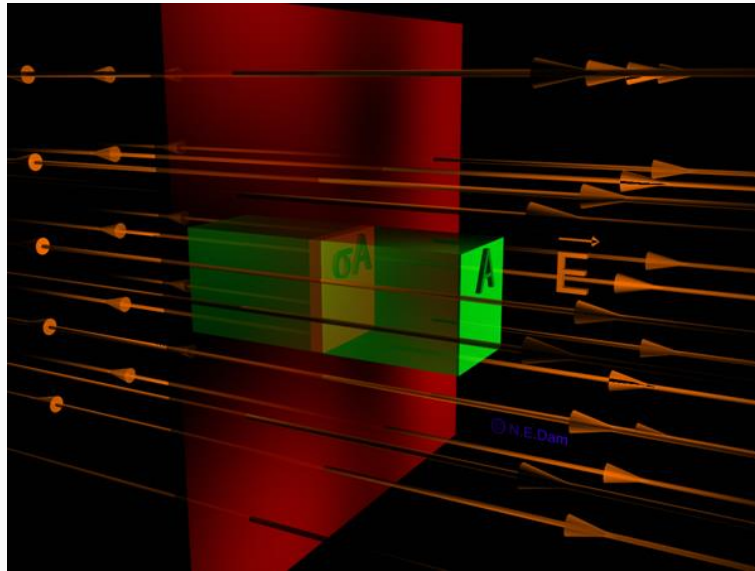


Fig.3.6.6, N.E. Dam ©. The E-field outside a single layer.

The flux through each of the two end surfaces of the green box (the GS) is EA because the E-field vector is perpendicular to the end surfaces.

There are also 4 sides parallel with the E-field. The flux through each of these 4 sides is zero, because the E-field is parallel with the sides and no E-flux lines pass through the 4 sides. When we integrate below, we should in fact integrate over all the 6 sides of the green box (the GS). However, since we only get contributions to the flux from the two end surfaces, the integral is reduced to cover only the two end surfaces indicated as “2A” in the integral. Then the total flux becomes

$$\Phi_E = \oint_{GS} \vec{E} \cdot d\vec{A} = \oint_{2A} E dA = E \oint_{2A} dA = E 2A = \frac{\sigma A}{\epsilon_0} \quad \text{eq. 3.6.9} \Rightarrow$$

$$E = \frac{\sigma}{2\epsilon_0} \quad \text{eq. 3.6.10}$$

This was obtained in a much more complicated manner by superposition of Coulomb fields in chapter 2. The magnitude is independent of the distance from the sheet because the sheet is infinite large.

Field inside a double layer of charge

We want to find the E-field in between an infinite sheet of positive charge and an infinite sheet of negative charge. The two sheets could be made of a conducting material. However, the two sheets could also be made of an insulating material with static induced charge. The so-called double layer of charge is shown in fig.3.6.7.

We assume that the two sheets are very close together. Then all the charge accumulates at the opposing faces shown in fig.3.6.7 since opposite charges attract. There will be no charge on the outer surfaces. The GS is shown as the green box enclosing the charge σA .

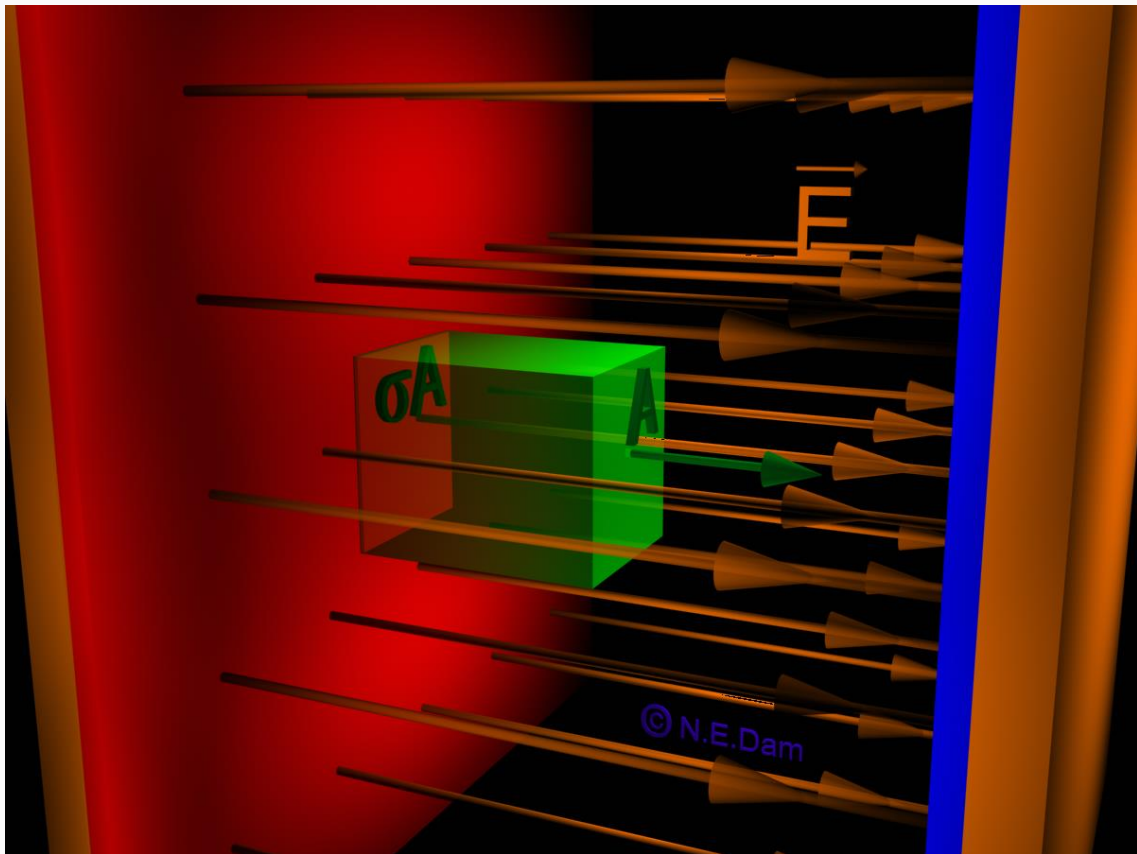


Fig.3.6.7, N.E. Dam ©. The E-field inside a double layer may be calculated using the concept of electric flux and Gauss's law.

The flux through the right end of the green box (the GS) is EA . The flux through the left end of the green box is zero, since the E-field is zero on the left side of the (red) positive charged sheet. All flux lines are to the right towards the (blue) negative charged sheet.

The flux is zero through the four other faces which are parallel with the E-field. Hence, the integral over the 6 sides of green box (GS) may be reduced to cover the right end surface (A) of the green box. The total flux through the GS becomes:

$$\Phi_E = \oint_{GS} \vec{E} \cdot d\vec{A} = \oint_A E dA =$$

$$E \oint_A dA = E A = \frac{\sigma A}{\epsilon_0} \quad \text{eq. 3.6.11} \Rightarrow$$

$$E = \frac{\sigma}{\epsilon_0} \quad \text{eq. 3.6.12}$$

This is twice as much as the E-field outside a single layer of charge.

If the two layers in the double layer are close together relative to the extension of the layers, the E-field outside the double layer is almost zero.

3.7 Charge distribution on conductors

Conductor with a cavity

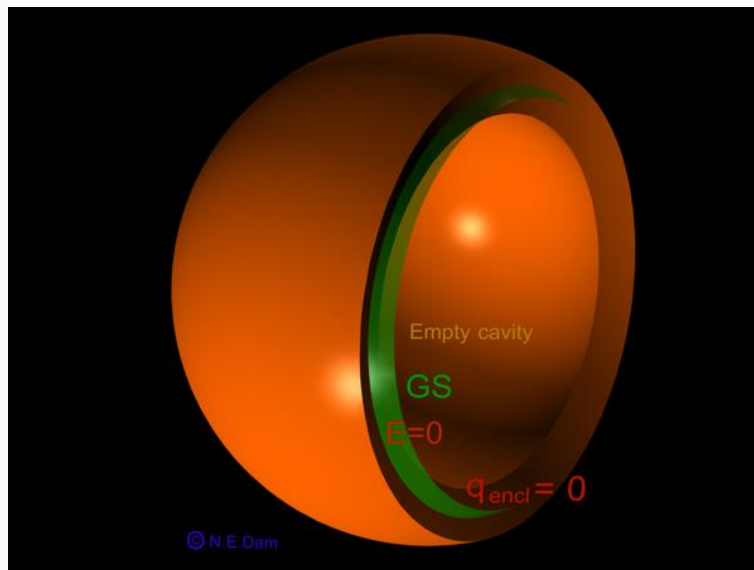


Fig.3.7.1, N.E.Dam ©. A conducting material with a cavity inside. In order to see the inner side of the cavity, the right part of the metal object is not shown.

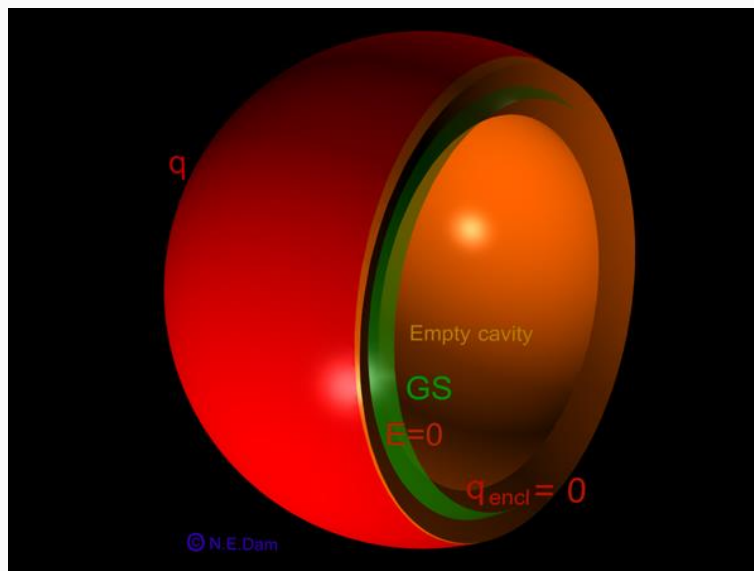


Fig.3.7.2, N.E. Dam ©. Charge applied to the outer surface will not change the charge inside the cavity.

Fig.3.7.1 shows a conducting material with a cavity inside. The situation is electrostatic. As mentioned earlier, the E-field inside the conducting material is zero. If we place a green GS inside the conducting material, the left side of Gauss's law is zero. Hence the right side of the same law must also be zero meaning that the enclosed charge is zero.

Fig.3.7.2 shows the same conducting material with a cavity inside. Now we have applied the charge q on the outer surface. Because of the electrostatic condition, the E-field inside the conductor is still zero. Again Gauss's law implies, that the enclosed

charge is zero. *No matter how we change the charge q on the outer surface, the enclosed charge inside the conductor and inside the cavity will remain zero.* This is in fact the Faraday cage, which will be shown in fig.3.7.14.

Fig.3.7.3 shows another situation. At first, we have no charge on the outer surface as shown in fig.3.7.1. Then we place a charge $+q$ inside the cavity. We imagine, that q is positive and therefore it is red. Because of the electrostatic condition, the E -field inside the conductor is still zero. Again Gauss's law implies, that the total enclosed charge inside the GS is zero. In this situation it implies that inside the GS there must be a (blue) charge $-q$ in addition to the charge $+q$. This negative charge is transferred from the outer surface to the inner surface of the cavity. It causes a deficit of negative charge on the outer surface which is the same as a positive excess charge on the outer surface. That is why we get the charge $+q$ on the outer surface.

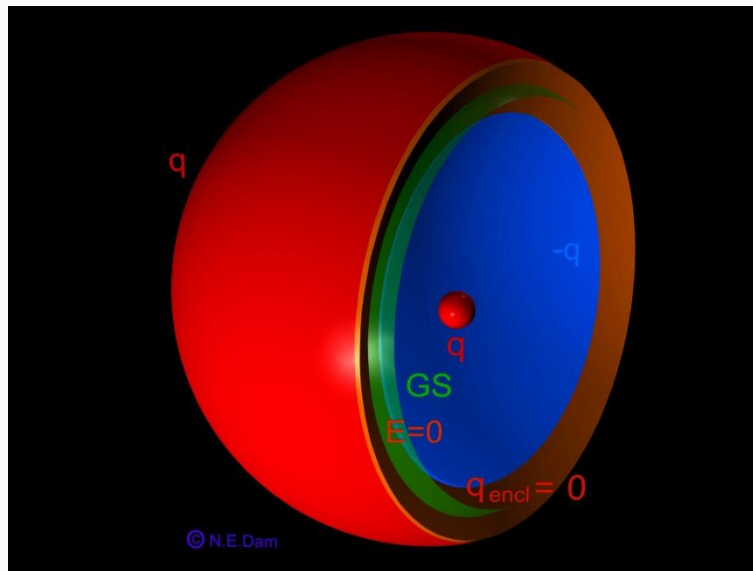


Fig.3.7.3, N.E. Dam ©. Introducing a charge $+q$ into the cavity will cause the negative charge $-q$ on the inner surface of the cavity and the charge $+q$ on the outer surface of the conductor.

The above argumentation would also be true if the charge q placed in the cavity was negative. Problem 3.1 illustrates how to calculate the charge on a conducting material with a cavity inside.

The above explanation of fig.3.7.1 and fig.3.7.3 may be used to explain the Faraday experiment in fig.3.7.7-10 and the Van der Graaf generator in 3.7.11-13. Charge transferred to a cavity inside a conductor will cause the same amount of charge on the outer surface of the conductor.

A more detailed explanation of fig.3.7.3.

Fig.3.7.4-6 shows some 2D sketches of what happens just before the electrostatic equilibrium is reached.

Fig.3.7.4 shows the positive charge q in the centre of the cavity. Initially, q emits an electric field directed radially away from q . This field also penetrates the metal, i.e. there is an electric field $\vec{E}_1$ inside the metal and also an electric field outside the metal. The electric field $\vec{E}_1$ inside the metal causes an electric force $\vec{F}_E$ on the (blue) electron shown in the surface. This electron thereby gets a drift velocity directed inward and stops when it reaches the inner surface, where negative charge $q_2(t)$ builds up. At the same time, positive charge builds up on the outer surface when electrons disappear from here. This results in an additional electric field $\vec{E}_2$ directed from the positive outer surface to the negative inner surface. The field $\vec{E}_2$ is oppositely directed to the field $\vec{E}_1$ from q and thus reduces the total electric field inside the metal, see fig.3.7.5.

The negative charge $q_2(t)$ on the inner side becomes more and more negative as more and more electrons migrate from the outer surface to the inner surface. Finally, the negative charge on the inner surface has grown to $q_2(t) = -q$ and $\vec{E}_2$ has become as large as $\vec{E}_1$, but oppositely directed, i.e. the field inside the metal has become zero. If we place a (green) Gaussian surface inside the metal as shown in fig.3.7.3, then the enclosed charge is $q_{\text{encl}} = q + (-q) = 0$. Thus it follows from Gauss's law that the E-field on the Gaussian surface inside the metal has become zero, see fig.3.7.6. When the E-field becomes zero, the migration of electrons from the outer surface towards the inner surface stops. Electrostatic equilibrium is achieved.

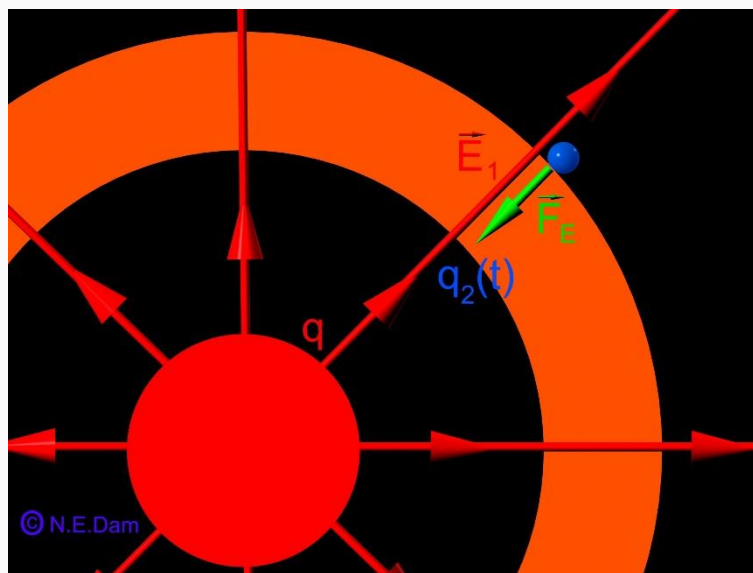


Fig.3.7.4, N.E. Dam ©. Introducing a (red) charge $+q$ into the cavity will cause a negative electron on the outer surface to migrate through the (orange) metal towards the inner surface of the metal.

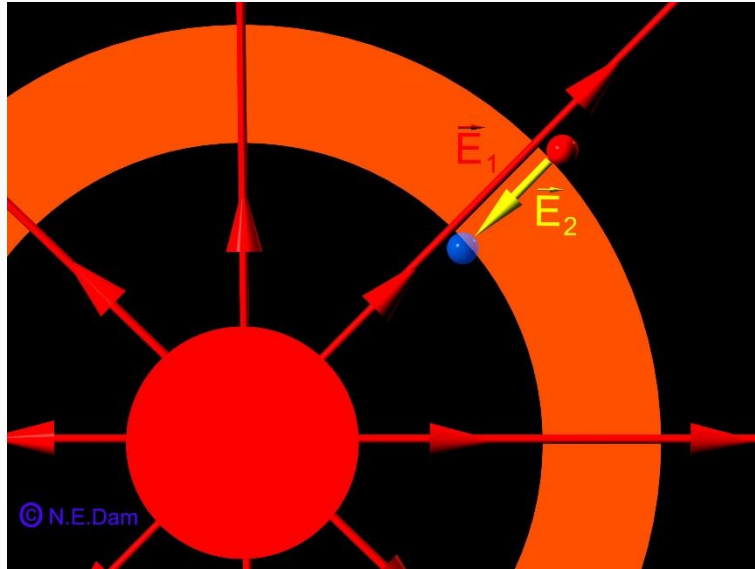


Fig.3.7.5, N.E. Dam ©. The appearance of negative charge on the inner surface and positive charge on the outer surface generates an additional electric field $\vec{E}_2$ inside the metal which counteracts the original field $\vec{E}_1$ inside the metal.

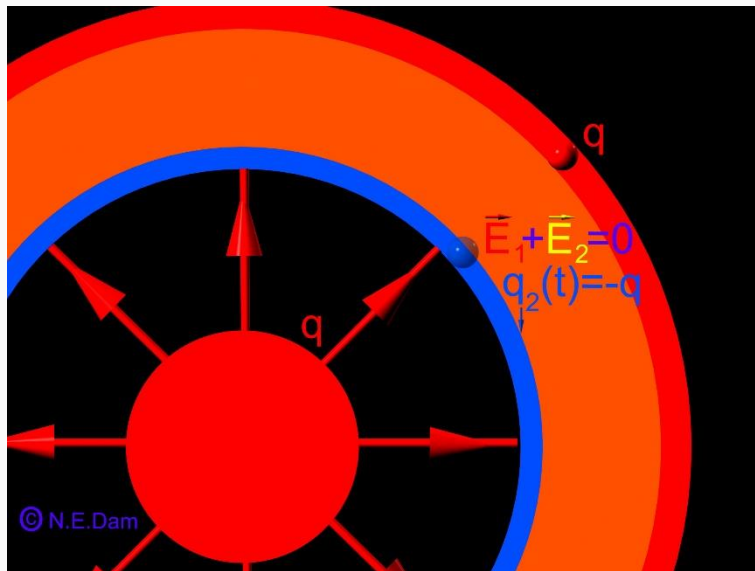


Fig.3.7.6, N.E. Dam ©. When the negative charge on the inner surface becomes $-q$, then the enclosed charge becomes zero for a Gaussian surface inside the metal and the field inside the metal becomes zero.

The Faraday experiment

Fig.3.7.7-10 shows the Faraday experiment. A positively charged metal sphere is suspended by an insulating thread and lowered into a metal bucket covered with a lid.

A negative charge is now induced on the inner surface as shown in fig.3.7.8 and in fig.3.7.9. The outer surface becomes positively charged, and we have exactly the same situation as in fig.3.7.3. In fig.3.7.9, we see a green GS inside the conducting material of the bucket. The E-field inside the material is zero. If we place a GS inside the material, the left side of Gauss's law is zero. Hence the right side of the same law must also be zero meaning that the enclosed charge is zero.

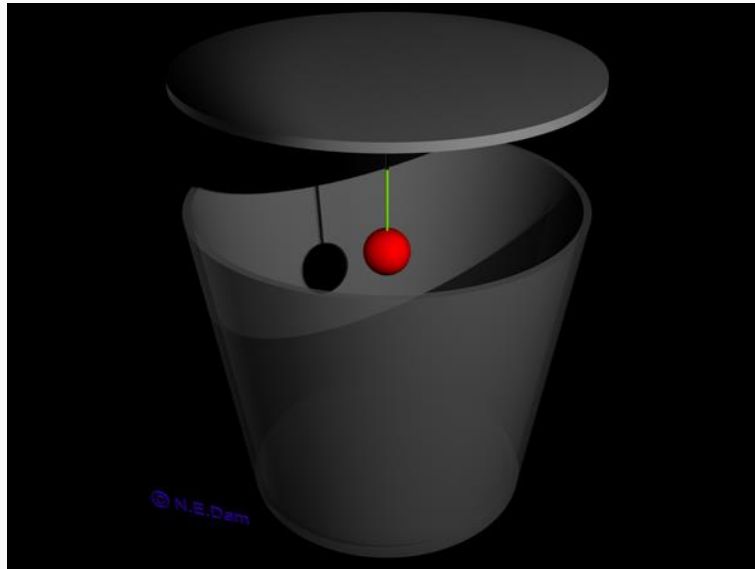


Fig.3.7.7, N.E. Dam ©. In the Faraday experiment a positively charged metal sphere is suspended by an insulating thread and lowered into a metal bucket.

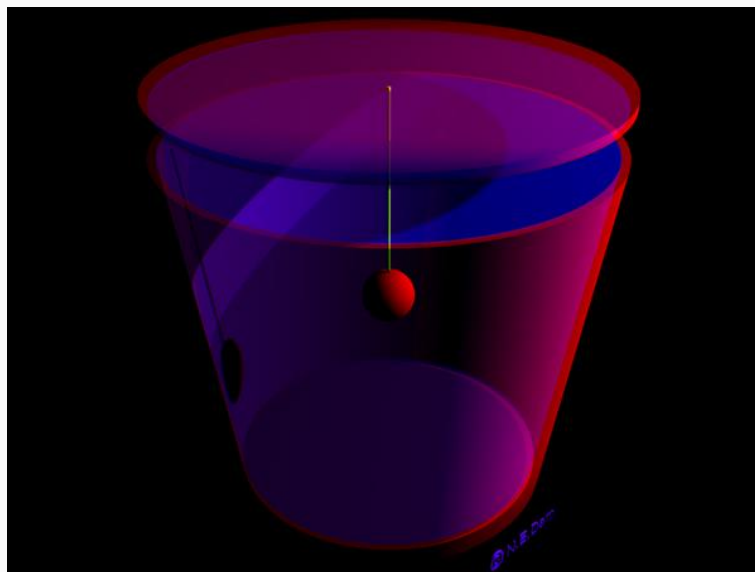


Fig.3.7.8, N.E. Dam ©. A negative charge is now induced on the inner surface and a positive charge on the outer surface.

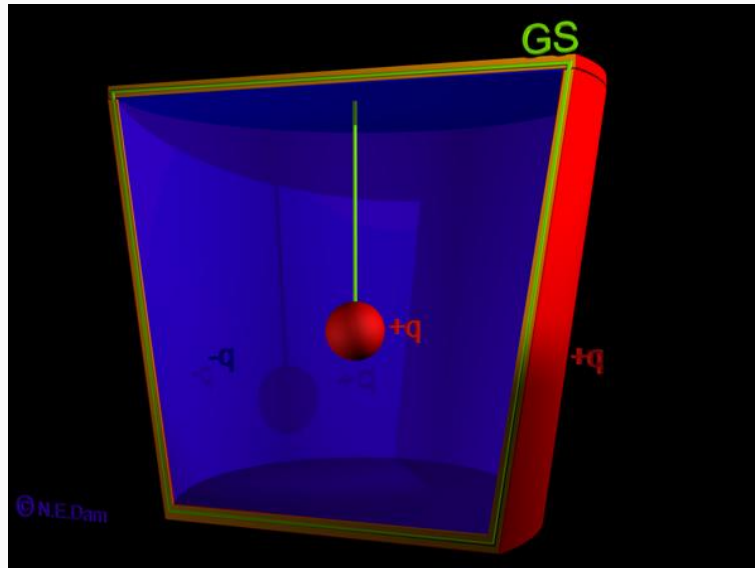


Fig.3.7.9, N.E. Dam ©. The cut away shows the (blue) negative charge inside and the (red) positive charge outside the bucket. The (orange) bucket metal between the inner and the outer surface is neutral.

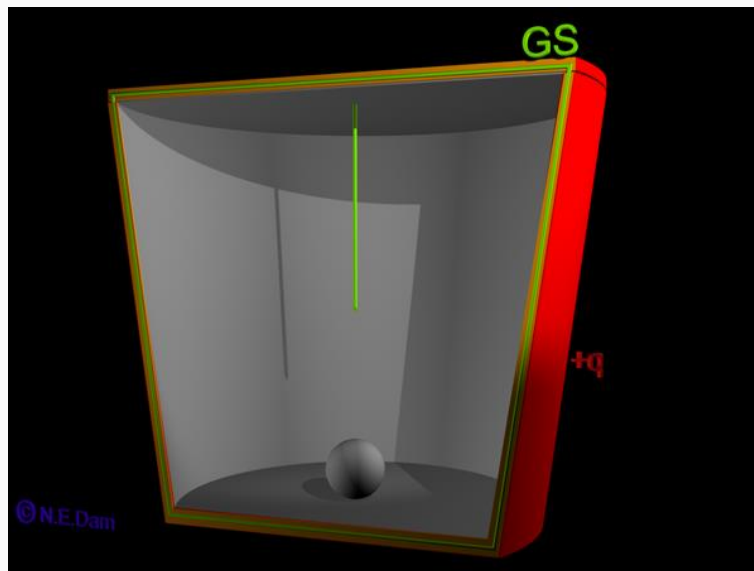


Fig.3.7.10, N.E. Dam ©. The metal sphere drops down and neutralizes the negative charge on the inner surface of the bucket.

If the thread breaks, the sphere falls down and becomes a part of the inner surface of the cavity as shown in fig.3.7.10. The positive charge on the sphere cancels the negative charge on the inner surface of the bucket. Now the inner surface has zero charge. The overall result is that the positive charge on the sphere has been transferred to the outer surface of the bucket. The charge cannot disappear. This experiment was carried out by Faraday in the 1800's and the experiment confirms Gauss's law.

The Van der Graaf generator

Fig.3.7.11 shows a Van der Graaf electrostatic generator, which can produce a high charge on the conducting shell and a high field from this shell. The conducting shell is at the top. A rubber band rotates and transports positive charge from the bottom to the top, where it is introduced into the shells inner surface. The two plates on both sides of the rubber band are made of an insulating material, e.g. acrylic plastic. The sides keep the two rollers in position.

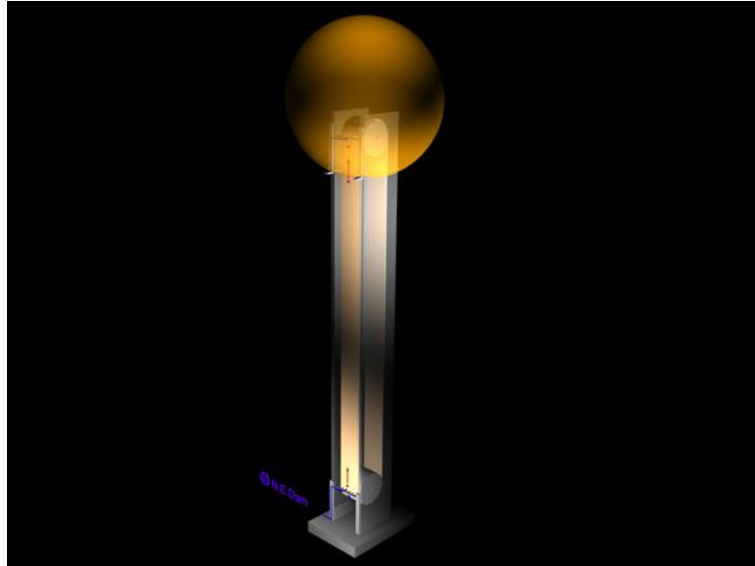


Fig.3.7.11, N.E. Dam ©. The Van der Graaf generator.

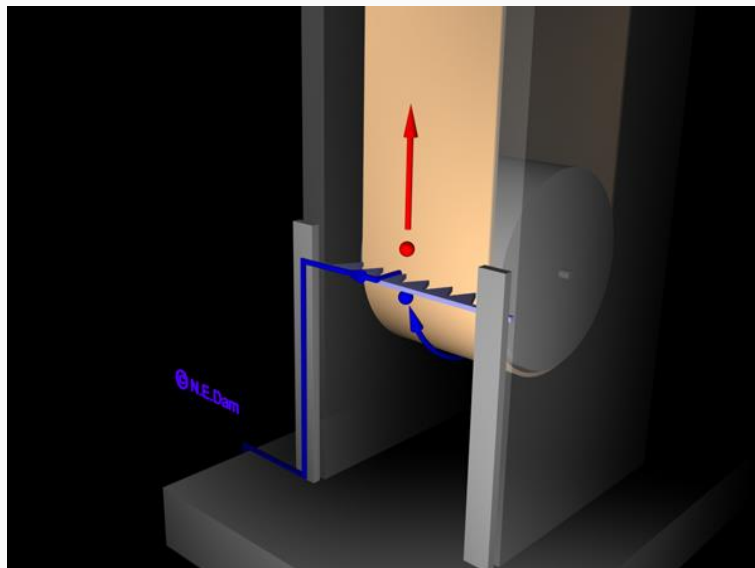


Fig.3.7.12, N.E. Dam ©. Negative charge is stripped of the surface of the rubber band when passing by the metal comb placed at the lower roller.

Fig.3.7.12 shows how a single electron is stripped off the surface of the rubber band when passing by the metal comb placed on the lower roller. The comb has spikes and sharp corners. In chapter 5 we will see, that we get a very strong E-field close to sharp

corners. This strong E-field causes a strong electric force on the electrons in the surface of the rubber band and some of these electrons are pulled out of the surface by this force. These electrons are conducted to the ground by a wire connected to the comb. The electron flows through the metal connector to the foot of the generator and continues down to the ground. The deficit of an electron on the rubber band is equivalent to a positive excess charge on the rubber band. This charge is now transported to the top.

Fig.3.7.13 shows the situation at the top. The hollow shell is equivalent to the cavity in the conductor shown in fig.3.7.1. When a positive charge is inserted into the cavity in fig.3.7.3, it corresponds to the situation, when positive charge on the rubber band is transported into the innerside of the shell as shown in fig.3.7.13. A negative electron from the outer surface of the shell will now flow to the inner side of the shell, where it may cancel the positive charge at the comb which is placed on the upper roller inside the hollow shell. The deficit of an electron at the outer surface is equivalent to a positive excess charge on the outer surface. The situation is just like the situation in fig.3.7.3. The charge introduced into the cavity is an increasing function of time $q(t)$. As mentioned, the charge on the outer surface will be the same as the charge introduced to the cavity. There will also be an amount of negative charge on the inner surface equal to minus the charge introduced to the cavity.

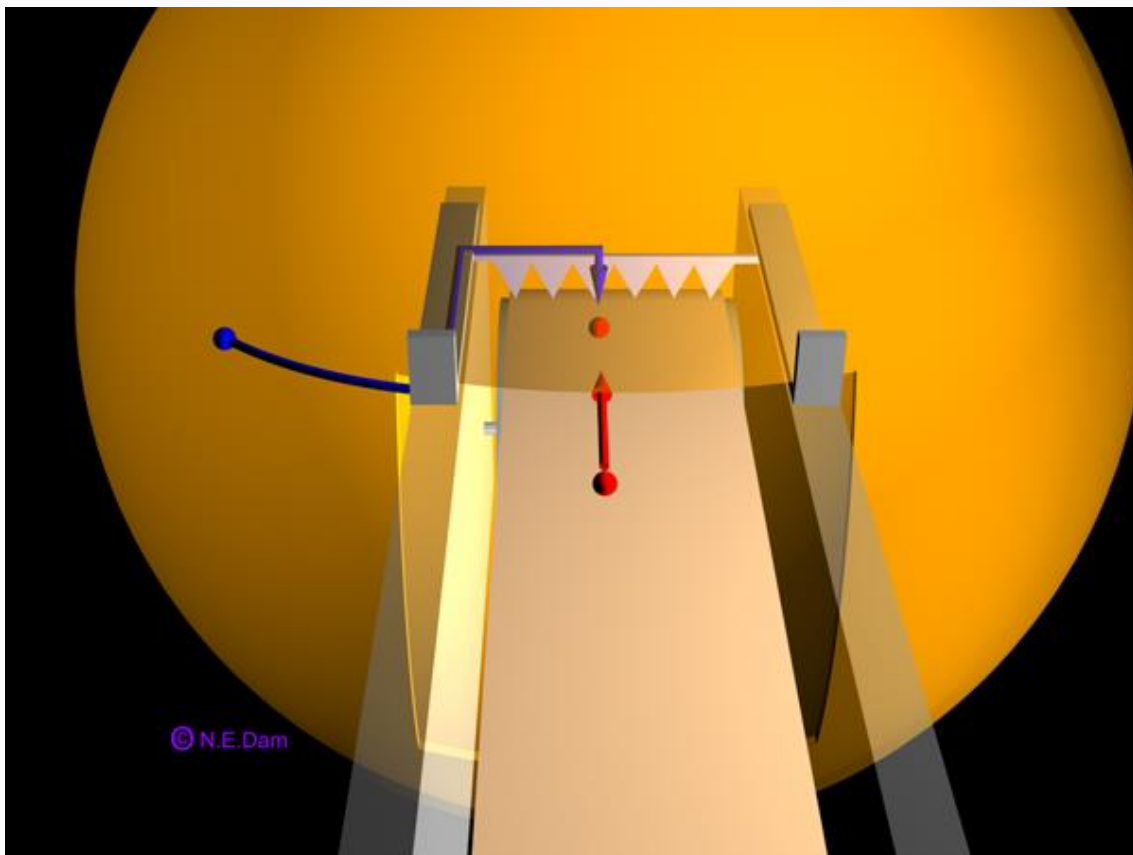


Fig.3.7.13, N.E. Dam ©. Positive charge on the rubber band is transported on the innerside of the shell at the top of the Van der Graaf.

The Faraday cage

Fig.3.7.14 shows a Faraday cage. In this case the Faraday cage is a metal box. It may be used as an electric shield for a person as shown in fig.3.7.15 or it may protect sensitive electronic equipment.

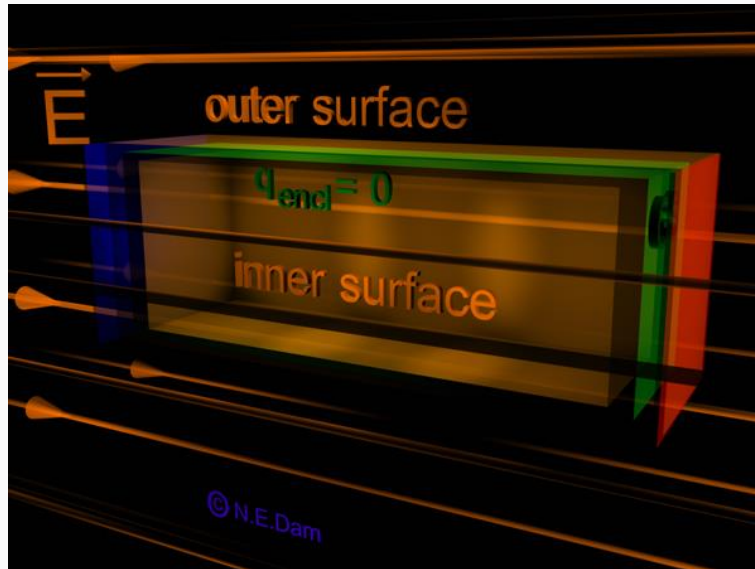


Fig.3.7.14, (N.E. Dam ©). A Faraday cage placed into an E-field.

The external field redistributes the electrons on the outer conductor surface as shown in the figure. The two end surfaces of the conducting metal box (the blue and the red end surface) are charged on the outer surface because of the E-field. Negative electrons are pulled by an electric force in the opposite direction of the E-field. Such electrons end up at the left end surface leaving a deficit of electrons (equivalent with a positive excess charge) at the right end surface.

The inner surface of the box is equivalent to the cavity shown in fig.3.7.1 and fig.3.7.2. If we use Gauss's law on the green GS, we see that the total enclosed charge is zero. If we have no charge inside the cavity or inside the box from the beginning, there will be no charge later on. No matter how much we change the outside field and the charge on the outer surface, the charge inside the cavity or inside the box will remain unchanged (zero).

Fig.3.7.15 shows a person standing inside a Faraday cage. The cage is connected to the ground, and we see lightning from the Van der Graaf generator to the cage. The person inside the Faraday cage is protected against the very strong electric field between the top of the Van der Graaf and the grounded Faraday cage. If we are inside a car we will be protected against lightning, because the car acts as a Faraday cage isolated from the ground, see the video www.youtube.com/watch?v=ve6XGKZxYxA.



*Fig.3.7.15, ref. www.tyrtle.wordpress.com
Two very large Van der Graaf generators.*

3.8 Gauss's law in differential form

In section 3.4 we derived Gauss's law for the E-field

$$\Phi_E = \oint_{GS} \vec{E} \cdot d\vec{A} = \frac{Q_{encl}}{\epsilon_0} \quad \text{eq. 3.4.6}$$

This law is also denoted Gauss's law in integral form because it is based on a closed surface integral. In addition to this law, Gauss developed a method, the so-called *Gauss's theorem*, to transform the closed surface into a volume integral. Thereby it is possible to write Gauss's law in differential form. If one work with electromagnetic waves it may be more convenient to use the differential form.

Below we derive Gauss's law in differential form. First, we consider Gauss's theorem which enables us to transform the closed surface integral in Gauss's law into a volume integral. Then we use Gauss's theorem to derive Gauss's law in differential form.

Gauss's theorem (the divergence theorem)

In the following, we widely use the so-called Del operator $\vec{\nabla}$ symbolized by the so-called nabla sign ∇ . We define the so-called *divergence* for a vector field $\vec{v}(x, y, z, t)$ by

$$\vec{\nabla} \cdot \vec{v} \equiv \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\} \cdot \{v_x, v_y, v_z\} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \quad \text{eq. 3.8.1}$$

Note, that the divergence $\vec{\nabla} \cdot \vec{v}$ is a scalar product and therefore not a vector. Now we present Gauss's (divergence) theorem

$$\int_V \vec{\nabla} \cdot \vec{v} \, dV = \oint_A \vec{v} \cdot d\vec{A} \quad \text{eq. 3.8.2}$$

This requires explanation! We consider a volume V enclosed by a surface A , see fig.3.8.1. Gauss's theorem then states, that if we integrate the divergence of $\vec{v}$ over the volumen V (the left side of eq.3.8.2), we get the the same result as if we integrate the flux of $\vec{v}$ out of the surface A (the right side of eq.3.8.2).

The divergence over the volume is the same as the flux out of the volume.

More explanation follows:

The right side integral

We know, that $\vec{v} \cdot d\vec{A}$ is the v -flux through the surface area element dA . The right side integral is therefore a summation of the v -flux through all the surface area elements dA , which gives the total v -flux out of the closed surface A .

The left side integral is a summation of $\vec{\nabla} \cdot \vec{v} \, dV$ over the volumen.

$\vec{\nabla} \cdot \vec{v} \, dV$ is the divergence of $\vec{v}$ times the volumen element dV .

$\vec{\nabla} \cdot \vec{v} \, dV$ is the flux coming out of the volumen element dV .

If we take the sum of $\vec{\nabla} \cdot \vec{v} \, dV$ over all volume elements $dV_1, dV_2, \dots$, we get the total v -flux coming out of the total volume V , which is the same as the v -flux, that passes out of the closed surface A .

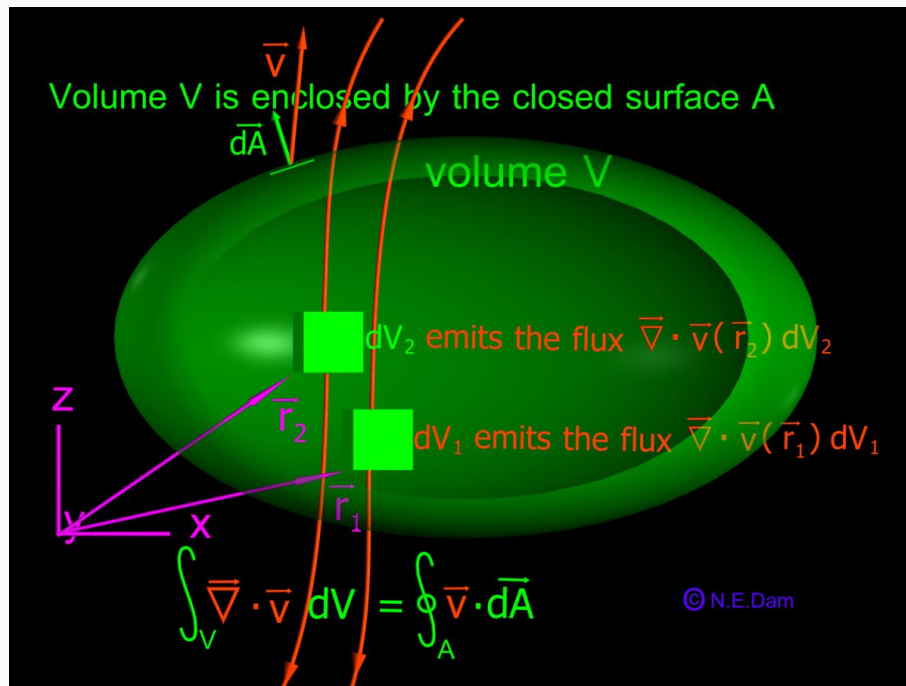


Fig.3.8.1, N.E. Dam ©. Gauss's theorem. The divergence over the volume V is the same as the flux out of the closed surface A , which encloses the volume.

As mentioned, $\vec{\nabla} \cdot \vec{v} dV$ is the flux coming out of the volumen element dV . This flux is proportional to the number of field lines or flux lines, that start within dV .

If $\vec{\nabla} \cdot \vec{v} dV$ is negative, it is the number of flux lines, that end within dV .

Since $\vec{\nabla} \cdot \vec{v} dV$ is the flux coming out of the volumen element dV we have that

The divergence $\vec{\nabla} \cdot \vec{v}$ at a point P is the flux per volume. We just need to multiply by dV to get the flux out of the volume element dV surrounding the point P .

Now we are ready to reformulate Gauss's law for the E-field.

Gauss's law for the E field

We repeat Gauss's law for the E field

$$\Phi_E = \oint_A \vec{E} \cdot d\vec{A} = \frac{Q_{\text{enclosed}}}{\epsilon_0} \quad \text{eq. 3.8.3}$$

Fig.3.8.2 shows Gauss's theorem for the volume V enclosed by the surface A

$$\int_V \vec{\nabla} \cdot \vec{E} dV = \oint_A \vec{E} \cdot d\vec{A} = \frac{Q_{\text{enclosed}}}{\epsilon_0} = \frac{1}{\epsilon_0} \int_V \rho dV \quad \text{eq. 3.8.4}$$

where $\rho = \rho(x,y,z)$ is the charge density at position (x,y,z) .

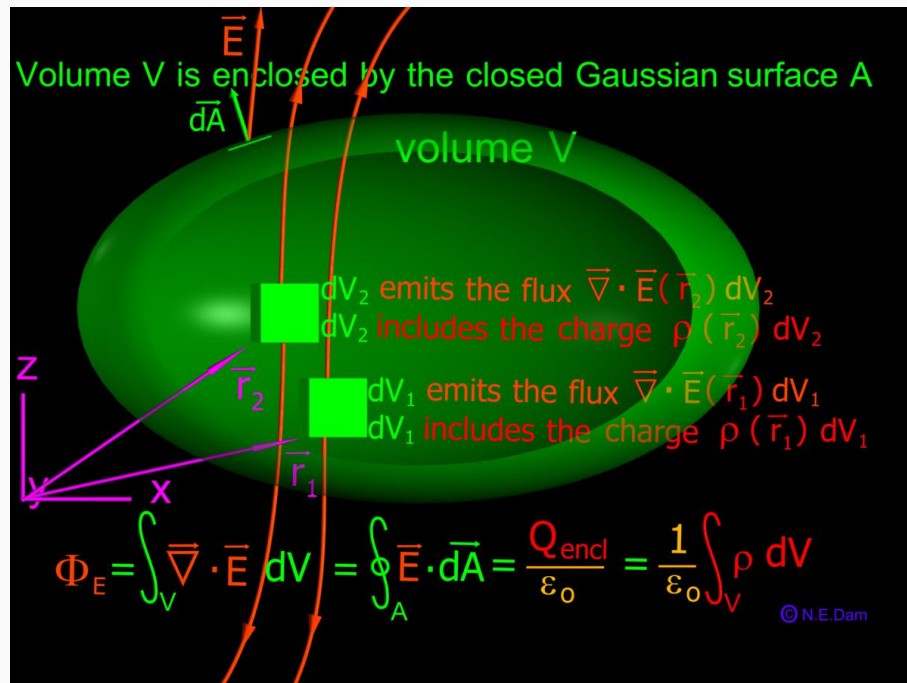


Fig.3.8.2, N.E. Dam ©. Flux out of a closed surface A which encloses the volume V.

Explanation follows:

The position of the volume elements is referred relative to a coordinate system. E.g. volume dV_1 has the position $\vec{r}_1$. At this position we have:

The E-field is $\vec{E}(\vec{r}_1)$

The divergence of the E-field is $\vec{\nabla} \cdot \vec{E}(\vec{r}_1)$, which is $\vec{\nabla} \cdot \vec{E}(\vec{r})$ taken at position $\vec{r}_1$.

The flux out of dV_1 is the divergence times the volume element: $\vec{\nabla} \cdot \vec{E}(\vec{r}_1) dV_1$

Likewise the flux out of dV_2 at position $\vec{r}_2$ is $\vec{\nabla} \cdot \vec{E}(\vec{r}_2) dV_2$

Now we take the sum of the flux from all the volume elements and obtain the total flux out of the total volume V (the left side integral in eq.3.8.4). This flux passes out of the closed surface A enclosing the volume V. The next integral in eq.3.8.4 is the well known calculation of the flux out of the closed surface A.

We see why these two integrals are identical according to Gauss's theorem.

According to Gauss's law, the flux out of V or the flux passing out of the closed surface A equals the enclosed charge divided by ϵ_0 .

The enclosed charge may be calculated as the volume integral of the volume charge density ρ , which will be explained below:

Consider the red text at the volume elements in fig.3.8.2.

At position $\vec{r}_1$ the volume charge density is $\rho(\vec{r}_1)$. Hence the amount of charge inside the volume element dV_1 is $\rho(\vec{r}_1) dV_1$.

Likewise the amount of charge inside the volume element dV_2 is $\rho(\vec{r}_2) dV_2$.

Now we take the sum of the charges from all the volume elements. Thereby we get the total charge within the volume V enclosed by the closed surface A. This is the right side integral in eq.3.8.4. If we compare with the left side integral in the same equation, we see that

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad \text{eq. 14.4.3}$$

which is *the differential form of Gauss's law for the E field*.

This equation is valid for any volume V, i.e. the volume V may be chosen freely.

$\vec{\nabla} \cdot \vec{E} \, dV$ is the divergence of the E-field times the volumen element dV .

$\vec{\nabla} \cdot \vec{E} \, dV$ is the E-flux out of dV . As we know, the flux is proportional to the number of flux lines.

A point P, where $\vec{\nabla} \cdot \vec{E} \, dV > 0$ is called a *source point*. This is a point where fluxlines start.

$\vec{\nabla} \cdot \vec{E} \, dV$ is the amount of E-flux, that flows out of dV .

A point P, where $\vec{\nabla} \cdot \vec{E} \, dV < 0$ is called a *drain point*. This is a point where fluxlines end.

$\vec{\nabla} \cdot \vec{E} \, dV$ is the amount of E-flux, that flows into dV .

The above remarks is for a normal E-field generated by + and – charges. If we consider the induced E-field $\vec{E}_i$, which is introduced in chapter 13, it has no source points. For such an E-field we have

$$\vec{\nabla} \cdot \vec{E}_i \, dV = 0 \quad \text{eq. 14.4.4}$$

Problems

3.1 Charge on a conducting material with a cavity inside

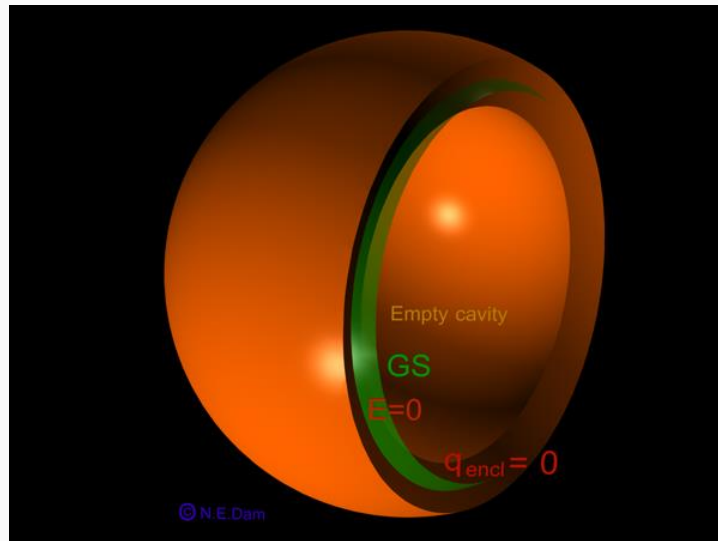


Fig.3.7.1, N.E. Dam ©. A conducting material with a cavity inside.

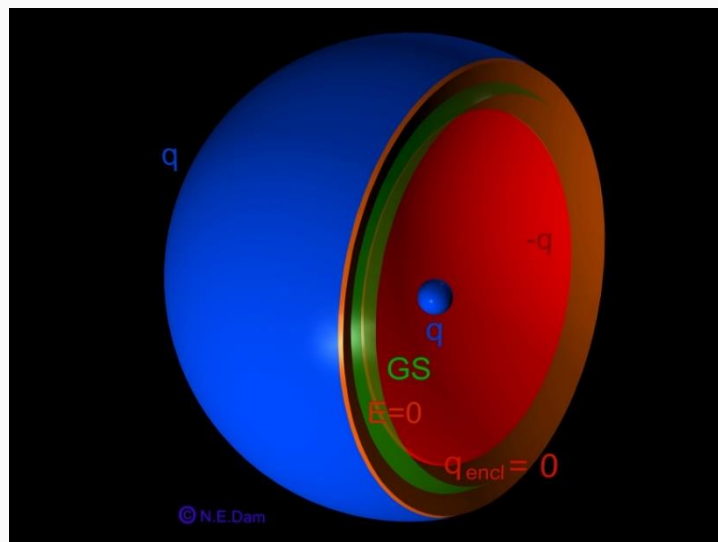


Fig.3.1, N.E. Dam ©. There is a negative charge inside the cavity.

To begin with, the conductor has no net charge outside or inside as in fig.3.7.1. Then we place a charge of $q = -4 \text{ nC}$ inside the cavity, see fig.3.1.

1. What happens right after we have placed q inside the cavity?
2. What is the charge on the inner surface of the cavity?
3. What is the charge on the outer surface on the conductor?
4. Now we add $q_{\text{extra}} = +8 \text{ nC}$ to the outer surface.
5. What is the charge on the inner surface of the cavity?
6. What is the charge on the outer surface of the conductor?

Test 3

Test 3A: Question 1-5, Test 3B: Question 6-9, Test 3C: Question

1. (25p) The definition of flux Φ . (NB! the flux per flux line is Φ_0)
 - a. Verbal definition of flux
 - b. The flux if we have N_{FL} flux lines through an area A
 - c. The flux if we have dN_{FL} flux lines through an area dA .
2. (20p) Consider dN_{FL} E-flux lines through an area dA . The E-flux per flux line is Φ_{E0} .
 - a. What is the E-flux, $d\Phi_E$, through dA ?
 - b. What is $d\Phi_E/dA$ and how is this related to the E-field magnitude?
- 3! (25p) The E-flux through a tilted area.
 - a. What is the E-flux $d\Phi_E$ through dA expressed as scalar product?
 - b. What is the area vector $\vec{dA}$?
 - c. The sign of the flux?
- 4i (10p) What is the flux Φ_E through a curved surface expressed as an integral?
- 5i (20p) Flux out of a closed surface.
 - a. What is the E-flux out of a closed surface A ?
 - b. When is the flux positive?
- 6! (15p) Gauss's law.
 - a. Please write Gauss's law
 - b. What is GS ?
 - c. What is q_{encl} ?
- 7! (25p) The E field outside a charged sphere.
 - a. Please use Gauss's law to calculate the E field outside a charged sphere (charge Q) in distance r from the center of the sphere.
 - b. Please make a small sketch.
- 8i (25p) The E field from a charged sheet.
 - a. Please use Gauss's law to calculate the E field from a charged sheet (area charge density σ).
 - b. Please make a small sketch.
9. (35p) Charge in a cavity in a conductor.
 - a. Please use Gauss's law and electric force to explain how the electrons move if we place a charge $q > 0$ inside a cavity in a conductor?
 - b. Please make a small sketch.

10i (50p) Gauss's law.

- a. Please make a small sketch with $q_1 > 0$ and $q_2 < 0$ inside the Gaussian Surface (GS) and $q_3 > 0$ outside the Gaussian Surface (GS)
- b. Please derive Gauss's law for the E-flux out of a closed surface GS ?
- c. What is q_{encl} ?

11. (50p). Gauss's law for the E-field in differential form.

- a. Please write the integral form of Gauss's law for the E-field again
- b. Express the enclosed charge inside the GS as a volumen integral over V using the charge density ρ
- c. Please use Gauss's theorem in connection with the E-field and derive Gauss's law for the E-field in differential form. Explain the terms and make a sketch.